Tilting for Artin-Schelter Gorenstein algebras of dimension one

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Contents

Introduction

Throughout this talk, let

- k : field
- $A := \bigoplus_{i \in \mathbb{N}} A_i$: $\mathbb{N}$-graded Noetherian k-algebra with $\dim_k A_i < \infty$
- $\mathsf{mod}^{\mathbb{Z}} A$: finitely generated $\mathbb{Z}$-graded right A-modules
- $\mathsf{proj}^{\mathbb{Z}} A$: finitely generated $\mathbb{Z}$-graded projective A-modules

Definition

(1) A is d-Iwanaga-Gorenstein if $\mathrm{inj.dim}(A_A) = \mathrm{inj.dim}({}_A A) = d < \infty$.

(2) The category of $\mathbb{Z}$-graded Cohen-Macaulay A-modules:

$$\mathsf{CM}^{\mathbb{Z}} A := \{X \in \mathsf{mod}^{\mathbb{Z}} A \mid \mathrm{Ext}^i_A(X, A) = 0 \quad \forall i > 0\}$$

(3) The projective stable category of $\mathsf{CM}^{\mathbb{Z}} A$:

$$\underline{\mathsf{CM}}^{\mathbb{Z}} A := \mathsf{CM}^{\mathbb{Z}} A / [\mathsf{proj}^{\mathbb{Z}} A]$$

If A is Iwanaga-Gorenstein, then $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ is a triangulated category by [Buchweitz, Happel].

Introduction

Let $\mathcal{C}$ be a Hom-finite algebraic triangulated category (e.g. $\mathcal{C} = \underline{\mathsf{CM}}^{\mathbb{Z}} A$).

Theorem [Rickard, Keller]

If $\mathcal{C}$ has a tilting object T i.e.

- $\mathrm{Hom}_{\mathcal{C}}(T, T[i]) = 0 \ \ \forall i \neq 0$
- the minimal thick subcategory of $\mathcal{C}$ containing T is $\mathcal{C}$

then $\mathcal{C} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathrm{End}_{\mathcal{C}}(T))$.

Question

For an Iwanaga-Gorenstein algebra A,

- When does $\underline{\mathsf{CM}}^{\mathbb{Z}} A$ have a tilting object T?
- What is $\mathrm{End}(T)$?

Today

Study this question for Artin-Schelter Gorenstein algebras.

Introduction

Results of tilting for Iwanaga-Gorenstein algebras

(a) [Yamaura'13] : finite dimensional self-injective algebra
(b) [K'19, Lu-Zhu'21, K-Minamoto-Yamaura'22] : finite dimensional 1-IG algebra
(c) [I-Takahashi'13] : Quotient singularity
(d) [Mori-U'16] : Noncommutative quotient singularity

Theorem [Buchweitz-I-Yamaura'20]

Let A be a commutative Noetherian Gorenstein ring with Krull dimension 1 and $A_0 = k$. Let $p_A \in \mathbb{Z}$ be the <u>Gorenstein parameter</u> of A, i.e. $\mathrm{Ext}^1_A(k, A) \simeq k(p_A)$. Then the followings are equivalent

- $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ has a tilting object
- $p_A \leq 0$ or $\mathrm{gldim} A < \infty$

where $\mathsf{CM}_0^{\mathbb{Z}} A = \{X \in \mathsf{CM}^{\mathbb{Z}} A \mid X_\mathfrak{p} \in \mathsf{proj} A_\mathfrak{p} \ \ \forall \mathfrak{p} \in \mathsf{Spec}\, A \text{ s.t. } \mathrm{ht}\, \mathfrak{p} = 0\}$.

Our result

Recall that $A = \bigoplus_{i \in \mathbb{N}} A_i$ is $\mathbb{N}$-graded. Assume that

- A_0 is basic with a complete set of pairwise orthogonal idempotents

$$1_{A_0} = e_1 + e_2 + \cdots + e_n. \qquad \mathbb{I} := \{1, 2, \ldots, n\}$$

- $S_i := \mathrm{top}(e_i A_0)$: simple A-module
- $S'_i := DS_i$: simple left A-module, where $D = \mathrm{Hom}_k(-, k)$

Definition

A is an Artin-Schelter Gorenstein algebra of dimension d if

- A is d-Iwanaga-Gorenstein (i.e. $\mathrm{inj.dim}(A_A) = \mathrm{inj.dim}({}_A A) = d < \infty$)
- there exist $\nu \in \mathfrak{S}_n$ and $p_i \in \mathbb{Z}$ $(i \in \mathbb{I})$ such that for each $i \in \mathbb{I}$

$$\mathrm{Ext}^j_A(S_i, A) \simeq \begin{cases} S'_{\nu(i)}(p_i) & j = d \\ 0 & \text{else.} \end{cases}$$

We call $(p_i)_{i \in \mathbb{I}}$ Gorenstein parameters and call ν the Nakayama permutation of A.

Our result

Example 1

Let R be an $\mathbb{N}$-graded commutative Noetherian Gorenstein k-algebra with $\dim R = d$. An $\mathbb{N}$-graded R-algebra A is called a Gorenstein R-order if

$$A_R \in \mathsf{CM}R \quad \text{and} \quad \mathrm{Hom}_R(A_A, R) \in \mathsf{proj}({}_A A).$$

A Gorenstein R-order is an AS-Gorenstein algebra of dimension d.

Example 2

Let $R = k[x]$ with $\deg x = 1$ and $\mathfrak{m} = (x)$. Then for $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b > 0$

$$A = \begin{pmatrix} R & \mathfrak{m}^a \\ \mathfrak{m}^b & R \end{pmatrix}$$

is a Gorenstein order. So A is an AS-Gorenstein algebra of dimension 1.
Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\nu = (1\ 2)$ and $(p_1, p_2) = (1 - b,\ 1 - a)$.

Our result

Let A be an AS Gorenstein algebra of dimension 1.

- $\mathsf{qgr}A := \mathsf{mod}^{\mathbb{Z}}A/\mathsf{mod}_0^{\mathbb{Z}}A$, where $\mathsf{mod}_0^{\mathbb{Z}}A = \{M \mid M \text{ has finite length}\}$
- $Q := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{qgr}A}(A, A(i))$: graded total quotient ring of A.

If A is a Gorenstein R-order with $\dim R = 1$, then $Q = A \otimes_R (H^{-1}R)$ where $H =$\{homogeneous nonzero divisors of R\}.

Definition

$$\mathsf{CM}_0^{\mathbb{Z}}A := \{M \in \mathsf{CM}^{\mathbb{Z}}A \mid M \otimes_A Q \in \mathsf{proj}^{\mathbb{Z}}Q\}.$$

Theorem [Auslander-Reiten, I-Takahashi, IKU]

Let A be an AS Gorenstein algebra of dimension 1. Then there exists an invertible A-bimodule ω such that $(-) \otimes_R \omega$ induces a Serre functor

$$(-) \otimes_R \omega : \underline{\mathsf{CM}}_0^{\mathbb{Z}}A \longrightarrow \underline{\mathsf{CM}}_0^{\mathbb{Z}}A.$$

Our result

Main Theorem [IKU]

Let A be an AS Gorenstein algebra of dimension 1 with Gorenstein parameters $(p_i)_{i\in\mathbb{I}}$. Assume that A is ring-indecomposable and $\operatorname{gldim} A_0 < \infty$.

(a) $\exists N > 0$ such that $V = \bigoplus_{i=1}^{N} A(i)_{\geq 0}$ is a silting object of $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$.

(b) If $p_i \leq 0$ for any $i \in \mathbb{I}$, then V is tilting.

(c) In the case (b), we have a description of $\operatorname{End}(V)$ by using A, Q and $(p_i)_{i\in\mathbb{I}}$.

(d) The following statements are equivalent.

  (i) $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$ has a tilting object

  (ii) $\sum_{i\in\mathbb{I}} p_i \leq 0$ or $\operatorname{gldim} A < \infty$

Example

Let $R = k[x]$ with deg $x = 1$, $\mathfrak{m} = (x) = Rx$. Let $\mathbb{I} := \{1, 2, \dots, n\}$.

Definition

A Gorenstein tiled order is an R-subalgebra A of $\mathrm{M}_n(R)$ of the form

$$A = \begin{bmatrix} R & \mathfrak{m}^{m(1,2)} & \cdots & \mathfrak{m}^{m(1,n)} \\ \mathfrak{m}^{m(2,1)} & R & \cdots & \mathfrak{m}^{m(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{m}^{m(n,1)} & \mathfrak{m}^{m(n,2)} & \cdots & R \end{bmatrix}$$

for some $m(i,j) \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Hom}_R(A_A, R) \in \mathsf{proj}({}_A A)$.

Example

For $a, b, p, q, r \in \mathbb{Z}_{\geq 0}$ with $a + b > 0,\ p + q + r > 0$,

$$\begin{pmatrix} R & \mathfrak{m}^a \\ \mathfrak{m}^b & R \end{pmatrix}, \quad \begin{pmatrix} R & \mathfrak{m}^p & \mathfrak{m}^{p+q} \\ \mathfrak{m}^{q+r} & R & \mathfrak{m}^q \\ \mathfrak{m}^r & \mathfrak{m}^{r+p} & R \end{pmatrix}$$

are Gorenstein tiled orders.

Example

Proposition

(1) { Gorensten tiled order } $\subset$ { Gorenstein R-order } $\subset$ { ASG of dimension 1 }

(2) The Nakayama permutation ν satisfies

$$m(\nu(i), j) + m(j, i) = m(\nu(i), i) \quad \text{for each } i, j \in \mathbb{I}.$$

(3) The Gorenstein parameters are $p_i = 1 - m(\nu(i), i)$.

Example

For $p, q, r \in \mathbb{Z}_{\geq 0}$ with $p + q + r > 0$

$$A = \begin{pmatrix} R & \mathfrak{m}^p & \mathfrak{m}^{p+q} \\ \mathfrak{m}^{q+r} & R & \mathfrak{m}^q \\ \mathfrak{m}^r & \mathfrak{m}^{r+p} & R \end{pmatrix}.$$

- $\nu = (1\ 2\ 3)$
- $p_1 = 1 - q - r, \qquad p_2 = 1 - r - p, \qquad p_3 = 1 - p - q$

Example

Let A be a Gorenstein tiled order with the Gorenstein parameters $(p_i)_{i\in\mathbb{I}}$ and the Nakayama permutation ν.

Definition

(1) For an A-module $M = [\mathfrak{m}^{\ell_1}\ \mathfrak{m}^{\ell_2}\ \cdots\ \mathfrak{m}^{\ell_n}]$, let

$$\mathrm{v}(M) := (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{Z}^n.$$

(2) For $v, w \in \mathbb{Z}^n$, let $v \le w :\Longleftrightarrow v_i \le w_i$ for each $i \in \mathbb{I}$.

(3) We have a finite poset $(\mathbb{V}_A, \le)$, where

$$\mathbb{V}_A := \{\mathrm{v}(e_i A(j)_{\ge 0}) \mid i \in \mathbb{I},\ 1 \le j \le -p_{\nu^{-1}(i)}\} \cup \{0\} \subset \mathbb{Z}^n$$

Theorem [IKU]

Assume that $p_i \le 0$ for each $i \in \mathbb{I}$. Let $V = \bigoplus_{i=1}^{N} A(i)_{\ge 0}$ be the tilting object of $\underline{\mathsf{CM}}_0^{\mathbb{Z}} A$. Then $\mathrm{End}(V)$ is Morita equivalent to the incidence algebra $k(\mathbb{V}_A^{\mathrm{op}})$ of the opposite poset of $(\mathbb{V}_A, \le)$.

Example

Example

For $a, b \in \mathbb{Z}$ with $a + b > 0$

$$A = \begin{pmatrix} R & \mathfrak{m}^a \\ \mathfrak{m}^b & R \end{pmatrix}.$$

Assume that $p_1, p_2 \leq 0$ ($\Leftrightarrow b, a \geq 1$). We have

$$\text{add } V = \text{add}\left\{ \bigoplus_{i=1}^{a-1} e_1 A(i)_{\geq 0} \oplus (R\ R) \oplus \bigoplus_{j=1}^{b-1} e_2 A(j)_{\geq 0} \right\},$$

$$\mathbb{V}_A = \{(0\ i), (0\ 0), (j\ 0) \mid 1 \leq i \leq a-1,\ 1 \leq j \leq b-1\}$$

and $\mathrm{End}(V)$ is Morita equivalent to $k(\mathbb{V}_A^{\mathrm{op}}) \simeq kQ$ for a quiver Q as follows

$$Q = \underbrace{\bullet \rightarrow \cdots \quad \cdots \rightarrow \bullet}_{a-1} \rightarrow \bullet \leftarrow \underbrace{\bullet \leftarrow \cdots \quad \cdots \leftarrow \bullet}_{b-1}$$

Example

For $p, q, r \in \mathbb{Z}_{\geq 0}$ with $p + q + r > 0$

$$A = \begin{pmatrix} R & \mathfrak{m}^p & \mathfrak{m}^{p+q} \\ \mathfrak{m}^{q+r} & R & \mathfrak{m}^q \\ \mathfrak{m}^r & \mathfrak{m}^{r+p} & R \end{pmatrix}.$$

Assume that $p_1, p_2, p_3 \leq 0$ ($\Leftrightarrow q + r,\ r + p,\ p + q \geq 1$). Then $\mathrm{End}(V)$ is Morita equivalent to $k(\mathbb{V}_A^{\mathrm{op}}) \simeq kQ/I$, where Q is as follows and I is generated by commutative relations: