Defining relations of 3-dimensional cubic AS-regular algebras whose point schemes are reducible

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Contents

Contents

- k: an algebraically closed field of characteristic 0.
- A: a connected graded algebra over k, finitely generated in degree 1.
  - $A = k \oplus A_1 \oplus A_2 \oplus \cdots$, $\quad k \cong A/A_{\geq 1}$: a graded right A-module.
  - $A = T(A_1)/(R)$: a quotient of the tensor algebra $T(A_1)$ of A_1.
  - If $\{x_1, \ldots, x_n\}$ is a basis for A_1, then $A = k\langle x_1, \ldots, x_n\rangle/(R)$.
- $\mathbb{P}^{n-1}$: the $n-1$ dimensional projective space over k $(n \geq 2)$.

Definition 1.1 (Artin-Schelter, 1987)

A connected graded algebra A is called a d-dimensional Artin-Schelter regular (AS-regular) algebra if

1. $\operatorname{gldim} A = d < \infty$,
2. $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & (i = d) \\ 0 & (i \neq d). \end{cases}$ (Gorenstein condition)

Remark

If A is commutative, then
A: d-dimensional AS-regular algebra $\Leftrightarrow A \cong k[x_1, \ldots, x_d]$.

- A: 0-dimensional AS-regular algebra $\iff A \cong k$.
- A: 1-dimensional AS-regular algebra $\iff A \cong k[x]$.
- A: 2-dimensional AS-regular algebra $\iff A$ is isomorphic to $k\langle x,y\rangle/(xy-\lambda yx)$ or $k\langle x,y\rangle/(xy-yx-x^2)$ where $0 \neq \lambda \in k$ ([Artin-Schelter, 1987]).

- ([Artin-Schelter, 1987]) Every 3-dimensional AS-regular algebra is isomorphic to one of the following algebras:

$$k\langle x,y,z\rangle/(f_1,f_2,f_3) \text{ or } k\langle x,y\rangle/(g_1,g_2)$$

where $f_1,f_2,f_3 \in k\langle x,y,z\rangle_2$ (quadratic) and $g_1,g_2 \in k\langle x,y\rangle_3$ (cubic).
- ([Artin-Tate-Van den Bergh, 1990]) Every 3-dimensional AS-regular algebra determines and is determined by a pair (E,σ).
  - $E=\mathbb{P}^2$ or E is a cubic curve in $\mathbb{P}^2$ (quadratic).
  - $E=\mathbb{P}^1\times\mathbb{P}^1$ or E is a curve of bidegree $(2,2)$ in $\mathbb{P}^1\times\mathbb{P}^1$ (cubic).
- ([Itaba-M., 2021], [Itaba-M., 2022], [M., 2021]) We give a complete list of defining relations f_1,f_2,f_3 and classify them up to graded algebra isomorphism and graded Morita equivalence.

- $d \geq 4$: Unknown in general.

Contents

Twisted superpotentials

Definition 2.1 ((Bocklandt-Schedler-Wemyss, 2010), (Mori-Smith, 2016))

Let $s \in \mathbb{N}^+$. Let V be a finite dimensional k-vector space.
Define a linear map $\phi : V^{\otimes s} \to V^{\otimes s}$ by
$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{s-1} \otimes v_s) := v_s \otimes v_1 \otimes \cdots \otimes v_{s-2} \otimes v_{s-1}$.

1. $w \in V^{\otimes s}$ is called a superpotential if $\phi(w) = w$.
2. $w \in V^{\otimes s}$ is called a twisted superpotential if

$$(\tau \otimes \mathrm{id}^{\otimes s-1})\phi(w) = w$$

   for some $\tau \in \mathrm{GL}(V)$.

3. The i-th derivation quotient algebra of $w \in V^{\otimes s}$ is defined by

$$\mathcal{D}(w, i) := T(V)/(\partial^i w)$$

   where $\partial^i w$ is the "i-th left partial derivatives" of w ($i \geq 1$).

Example 1

Let V be a k-vector space with basis $\{x, y, z\}$. Let $w = xyz + yzx + zxy - (xzy + yxz + zyx) \in k\langle x, y, z\rangle_3$. Since

$$\begin{aligned}\phi(w) &= zxy + xyz + yzx - (yxz + zyx + xzy) \\ &= xyz + yzx + zxy - (xzy + yxz + zyx) = w,\end{aligned}$$

$w = xyz + yzx + zxy - (xzy + yxz + zyx)$ is a superpotential.

$$\begin{aligned}w &= xyz + yzx + zxy - (xzy + yxz + zyx) \\ &= x(yz - zy) + y(zx - xz) + z(xy - yx) \\ &= x\partial_x w + y\partial_y w + z\partial_z w\end{aligned}$$

$\partial_x w$, $\partial_y w$, $\partial_z w$: the left partial derivatives of w w.r.t. x, y, z

$$\begin{aligned}\mathcal{D}(w, 1) &= k\langle x, y, z\rangle/(\partial_x w, \partial_y w, \partial_z w) \\ &= k\langle x, y, z\rangle/(yz - zy, zx - xz, xy - yx) = k[x, y, z].\end{aligned}$$

Example 2

Let V be a k-vector space with basis $\{x, y, z\}$. Let $w = xyz + yzx - zxy + xzy - yxz + zyx \in k\langle x, y, z\rangle_3$. Since

$$\begin{aligned}\phi(w) &= zxy + xyz - yzx + yxz - zyx + xzy \\ &= xyz - yzx + zxy + xzy + yxz - zyx \neq w,\end{aligned}$$

$w = xyz + yzx - zxy + xzy - yxz + zyx$ is not a superpotential. If we set $\tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{GL}_3(k)$, then

$$(\tau \otimes \mathrm{id} \otimes \mathrm{id})\phi(w) = xyz + yzx - zxy + xzy - yxz + zyx = w,$$

so w is a twisted superpotential. In this case,

$$\mathcal{D}(w, 1) = k\langle x, y, z\rangle/(yz + zy, zx - xz, xy - yx).$$

Theorem 2.2 (Dubois-Violette, 2007)

For every d-dimensional "m-Koszul" AS-regular algebra A of "Gorenstein parameter" ℓ, there exists a unique twisted superpotential $w \in V^{\otimes \ell}$ such that $A \cong \mathcal{D}(w, \ell - m)$.

Theorem 2.3 (Mori-Smith, 2016)

Let $w \in V^{\otimes \ell}$ be a twisted superpotential such that $A = \mathcal{D}(w, \ell - m)$ is a d-dimensional "m-Koszul" AS-regular algebra of Gorenstein parameter ℓ. Then A is "Calabi-Yau" if and only if $\phi(w) = (-1)^{d+1} w$

Remark

- Every 3-dimensional quadratic AS-regular algebra is a 3-dimensional 2-Koszul AS-regular algebra of Gorenstein parameter 3.
- Every 3-dimensional cubic AS-regular algebra is a 3-dimensional 3-Koszul AS-regular algebra of Gorenstein parameter 4.
- If $A = \mathcal{D}(w, 1)$ is a 3-dimensional AS-regular algebra, then A is "Calabi-Yau" if and only if w is a superpotential.

Classification of 3-dimensional AS-regular algebras

Theorem 2.4 (Mori-Smith, 2017)

Superpotentials w *such that* $\mathcal{D}(w,1)$ *are* 3*-dimensional quadratic AS-regular algebras are classified.*

Theorem 2.5 (Mori-Ueyama, 2019)

Superpotentials w *such that* $\mathcal{D}(w,1)$ *are* 3*-dimensional cubic AS-regular algebras are classified.*

Theorem 2.6 ((Itaba-M., 2021), (Itaba-M., 2022), (M., 2021))

Twisted superpotentials w *such that* $\mathcal{D}(w,1)$ *are* 3*-dimensional quadratic AS-regular algebras are classified.*

Our aim

Classify twisted superpotentials w such that $\mathcal{D}(w,1)$ are 3-dimensional cubic AS-regular algebras.

Contents

Geometric algebras

- V: a k-vector space with basis $\{x_1, \dots, x_n\}$ $(n \geq 2)$.
- $A = k\langle x_1, \dots, x_n \rangle / (g_1, \dots, g_s)$: a cubic algebra $(s \geq 1)$.
  - $g_1, \dots, g_s \in k\langle x_1, \dots, x_n \rangle_3$: homogeneous elements of degree 3.
- $\Gamma_A := \{(p, q, r) \in (\mathbb{P}^{n-1})^{\times 3} \mid g_1(p, q, r) = \cdots = g_s(p, q, r) = 0\}$.
- A pair (E, σ) is called a geometric pair if $E \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is a projective variety and σ is an automorphism of E satisfying $\pi_1 \sigma = \pi_2$ where $\pi_i : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is the i-th projection $(i = 1, 2)$.

Definition 3.1 ((M.-Saito, 2023), cf.(Mori, 2006))

A cubic algebra $A = k\langle x_1, \dots, x_n \rangle / (g_1, \dots, g_s)$ is called geometric if there exists a geometric pair (E, σ) such that
(G1) $\Gamma_A = \{(p, q, (\pi_2\sigma)(p, q)) \in (\mathbb{P}^{n-1})^{\times 3} \mid (p, q) \in E\}$,
(G2) $(g_1, \dots, g_s)_3 = \{f \in k\langle x_1, \dots, x_n \rangle_3 \mid f(p, q, (\pi_2\sigma)(p, q)) = 0,\ \forall (p, q) \in E\}$.
In this case, we write $A = \mathcal{A}(E, \sigma)$ and E is called the point scheme of A.

Example 1

Let $A = k\langle x, y\rangle/(xy^2 - y^2x, x^2y - yx^2)$ and $p = (p_1, p_2), q = (q_1, q_2), r = (r_1, r_2) \in \mathbb{P}^1$. Then

$$(xy^2 - y^2x)(p, q, r) = p_1q_2r_2 - p_2q_2r_1 = (p_1r_2 - p_2r_1)q_2,$$
$$(x^2y - yx^2)(p, q, r) = p_1q_1r_2 - p_2q_1r_1 = (p_1r_2 - p_2r_1)q_1.$$

It follows that

$$(p, q, r) \in \Gamma_A \Longleftrightarrow r = p.$$

We define an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$, denoted by ν, by $\nu(p, q) = (q, p)$. In this case, we have that

$$\Gamma_A = \{(p, q, (\pi_2\nu)(p, q)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid (p, q) \in \mathbb{P}^1 \times \mathbb{P}^1\}.$$

Example 2

Let $A = k\langle x, y\rangle/(xy^2 - y^2x, x^2y - yx^2)$ and $p, q, r \in \mathbb{P}^1$.
Since $(p, q, r) \in \Gamma_A \Longleftrightarrow r = p$, it is clear that

$$(xy^2 - y^2x, x^2y - yx^2)_3 \subset \{f \in k\langle x, y\rangle_3 \mid f(p, q, p) = 0, \ \ \forall p, q \in \mathbb{P}^1\}.$$

Conversely, let $g \in \{f \in k\langle x, y\rangle_3 \mid f(p, q, p) = 0, \ \ \forall p, q \in \mathbb{P}^1\}$ and write

$$g = a_1x^3 + a_2x^2y + a_3xyx + a_4yx^2 + a_5xy^2 + a_6yxy + a_7y^2x + a_8y^3.$$

If $p = q = (1, 0) \in \mathbb{P}^1$, then $a_1 = g(p, q, p) = 0$.
If $p = (1, 0), q = (0, 1) \in \mathbb{P}^1$, then $a_3 = g(p, q, p) = 0$.
Similarly, we have that $a_6 = a_8 = 0$. If $p = q = (1, \lambda) \in \mathbb{P}^1$ where $\lambda \neq 0$, then $(a_2 + a_4)\lambda + (a_5 + a_7)\lambda^2 = g(p, q, p) = 0$, so $a_2 + a_4 = a_5 + a_7 = 0$.
Therefore, we have that
$g = a_2(x^2y - yx^2) + a_5(xy^2 - y^2x) \in (xy^2 - y^2x, x^2y - yx^2)_3$. Hence,
$A = k\langle x, y\rangle/(xy^2 - y^2x, x^2y - yx^2) = \mathcal{A}(\mathbb{P}^1 \times \mathbb{P}^1, \nu)$ is geometric.

Theorem 3.2 (M.-Saito, 2023)

Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be geometric algebras.
Then $A \cong A'$ if and only if there exists an automorphism μ of $\mathbb{P}^{n-1}$ such that $\mu \times \mu$ restricts to an isomorphism $\mu \times \mu : E \to E'$ and

$$\begin{CD} E @>{\mu\times\mu}>> E' \\ @V{\sigma}VV @VV{\sigma'}V \\ E @>>{\mu\times\mu}> E' \end{CD}$$

commutes.

Remark

1. We say that E and E' are 2-equivalent if there exists $\mu \in \mathrm{Aut}_k \mathbb{P}^{n-1}$ such that $\mu \times \mu$ restricts to an isomorphism $\mu \times \mu : E \to E'$.
2. Let $A = \mathcal{A}(E, \sigma)$ be a geometric algebra. If E is 2-equivalent to E', then there exists $\sigma' \in \mathrm{Aut}_k E'$ such that $A \cong \mathcal{A}(E', \sigma')$.

Let A and A' be connected graded algebras. We say that A and A' are graded Morita equivalent if $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$ are equivalent.

Theorem 3.3 (M.-Saito, 2023)

Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be geometric algebras.
Then A and A' are graded Morita equivalent if and only if there exists a sequence of automorphisms μ_n of $\mathbb{P}^{n-1}$ such that $\mu_n \times \mu_{n+1}$ restricts to an isomorphism $\mu_n \times \mu_{n+1} : E \to E'$ and

$$\begin{array}{ccc} E & \xrightarrow{\mu_n \times \mu_{n+1}} & E' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma'} \\ E & \xrightarrow[\mu_{n+1} \times \mu_{n+2}]{} & E' \end{array}$$

commutes for every $n \in \mathbb{Z}$.

Contents

Theorem 4.1 (Artin-Tate-Van den Bergh, 1990)

Every 3-dimensional cubic AS-regular algebra A is geometric. Moreover, when we write $A = \mathcal{A}(E, \sigma)$, the point scheme E of A is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark

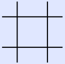
Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra where E is a curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then the point scheme E of A is one of the following types:

	reduced	nonreduced
reducible	S, T, S′, T′, FL	TWL
irreducible	NC, CC, EC (?)	WL

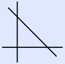
- ([Artin-Tate-Van den Bergh, 1991]) When $E =$  (Type TWL), the classification of A is completed.
- ([M.-Saito, 2023]) When E is either $\mathbb{P}^1 \times \mathbb{P}^1$,  (Type S), or  (Type T), the classification of A is completed.

Main Theorem 1 ([Itaba-M.-Saito])

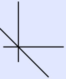
Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. Assume that E is either



(Type FL),



(Type S$'$),



(Type T$'$), or

(Type WL). For each case, we give a complete list of defining relations of A and classify them up to isomorphism and graded Morita equivalence in terms of their defining relations.

Remark

• For Type FL, S$'$ and T$'$, Theorem is proved by the following five steps: (1) Classify E up to 2-equivalence. (2) Find all $\sigma \in \mathrm{Aut}_k E$ satisfying $\pi_1\sigma = \pi_2$. (3) Calculate defining relations of $A = \mathcal{A}(E, \sigma)$ and a twisted superpotential w such that $A = \mathcal{D}(w, 1)$. (4) Check AS-regularity of A. (5) Classify them up to graded algebra isomorphism and graded Morita equivalence by using geometric conditions.

• For Type WL, we use the notions of "twisting system" and "twisted algebra" to prove Theorem.

TSPs of 3-dimensional cubic AS-regular algebras

Type	Potentials w	$\tau \in \mathrm{GL}_2(k)$
FL_1	$x^2y^2 - \alpha yx^2y + \alpha xy^2x + \alpha^2y^2x^2$	$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & -\alpha \end{pmatrix}$
FL_2	$-\alpha\beta x^4 + \beta xyxy + \beta yxyx - y^4$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
S'	$x^2y^2 + yx^2y - xy^2x + y^2x^2 - 2y^4$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
T'_1	$x^2y^2 - yx^2y - xy^2x + y^2x^2 - \alpha y^2xy + \alpha yxy^2$	$\begin{pmatrix} -1 & -\alpha \\ 0 & -1 \end{pmatrix}$
T'_2	$x^2y^2 - yx^2y - xy^2x + y^2x^2 + 2xy^3 + \alpha yxy^2 - \alpha y^2xy - 2y^3x + (\alpha+2)y^4$	$\begin{pmatrix} -1 & -(\alpha+2) \\ 0 & -1 \end{pmatrix}$
WL_1	$\alpha^4x^2y^2 + \alpha^2yx^2y + \alpha^2xy^2x + y^2x^2 - 2\alpha^3xyxy - 2\alpha yxyx$	$\begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix}$
WL_2	$x^2y^2 + yx^2y + xy^2x + y^2x^2 - 2xyxy - 2yxyx + 4yxy^2 - 4y^2xy + 2y^4$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

Classification up to graded algebra isomorphism

Type	defining relations ($\alpha, \beta \in k$)	condition
FL_1	$\begin{cases} xy^2 + \alpha y^2 x, \\ x^2 y - \alpha y x^2 \qquad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha, -\alpha^{-1}$
FL_2	$\begin{cases} -\alpha x^3 + yxy, \\ \beta xyx - y^3 \qquad (\alpha\beta \neq 0, \alpha \neq \beta) \end{cases}$	$(\alpha', \beta') = (\alpha, \beta)$ in $\mathbb{P}^1$
S'	$\begin{cases} xy^2 - y^2 x, \\ x^2 y + yx^2 - 2y^3 \end{cases}$	———
T'_1	$\begin{cases} xy^2 - y^2 x, \\ x^2 y - yx^2 + yxy - xy^2 \end{cases}$	———
T'_2	$\begin{cases} xy^2 - y^2 x + 2y^3, \qquad (\alpha \neq 0) \\ x^2 y - yx^2 - \alpha xy^2 + \alpha yxy + 2y^2 x - (\alpha + 2) y^3 \end{cases}$	$\alpha' = \alpha$
WL_1	$\begin{cases} \alpha^2 xy^2 + y^2 x - 2\alpha yxy, \\ \alpha^2 x^2 y + yx^2 - 2\alpha xyx \qquad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha^{\pm 1}$
WL_2	$\begin{cases} xy^2 + y^2 x - 2yxy, \\ x^2 y + yx^2 - 2xyx + 4xy^2 - 4yxy + 2y^3 \end{cases}$	———

Classification up to graded Morita equivalence

- Every Type FL_1 algebra is graded Morita equivalent to a Type FL_2 algebra.
- Every Type T'_2 algebra is graded Morita equivalent to a Type T'_1 algebra.
- Every Type WL_2 algebra is graded Morita equivalent to a Type WL_1 algebra.

Type	defining relations $(\alpha, \beta \in k)$	condition
FL	$\begin{cases} -\alpha x^3 + yxy, \\ \beta xyx - y^3 \qquad (\alpha\beta \neq 0, \alpha \neq \beta) \end{cases}$	$(\alpha', \beta') = (\alpha, \beta), (\beta, \alpha)$ in $\mathbb{P}^1$
S′	$\begin{cases} xy^2 - y^2x, \\ x^2y + yx^2 - 2y^3 \end{cases}$	———
T′	$\begin{cases} xy^2 - y^2x, \\ x^2y - yx^2 + yxy - xy^2 \end{cases}$	———
WL	$\begin{cases} xy^2 + y^2x - 2yxy, \\ x^2y + yx^2 - 2xyx \end{cases}$	———