Torsion-free classes of smooth projective curves of genus 0 and 1

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Overview

Today's topic

Study torsion-free classes of $\operatorname{coh} C$.

Here,

- torsion-free class = subcategory closed under sub. and ext.
- C: smooth projective curve ($\leftrightsquigarrow$ R: ring).
- $\operatorname{coh} C$: the category of coherent sheaves on C ($\leftrightsquigarrow$ $\operatorname{mod} R$)

More specifically, we classify torsion-free classes of $\operatorname{coh} C$ when:

- $C = \mathbb{P}^1$: the projective line (genus 0, easy case), and
- $C = E$: an elliptic curve (genus 1, main case).

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Contents

Throughout this talk,

- $\mathbb{k}$: an algebraically closed field.

Contents

The category coh C

C: smooth projective curve over $\Bbbk$ ($\leftrightsquigarrow$ R: ring).
Then coh C ($\leftrightsquigarrow$ mod R) is...

properties similar to mod $\Bbbk Q$:

- $\Bbbk$-linear abelian category,
- Hom-finite (i.e., $\dim_{\Bbbk} \mathrm{Hom}(\mathscr{F}, \mathscr{G}) < \infty$),
- hereditary (i.e., $\mathrm{Ext}^{>1} = 0$),

properties not similar to mod $\Bbbk Q$:

- noetherian but not of finite length,
- has no nonzero projective objects.

Theorem (Serre 1955)

If $C = \{f_1 = \cdots = f_m = 0\}$, $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$: homogeneous,

$$\mathrm{coh}\, C \simeq \mathrm{qgr}\, A := \mathrm{mod}^{\mathbb{Z}} A / \mathrm{mod}^{\mathbb{Z}}_{\mathrm{fd}} A, \quad A = \Bbbk[x_1, \ldots x_n]/(f_1, \ldots, f_m).$$

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The rough classification of C

Smooth projective curves C are roughly classified by $g_C \in \mathbb{Z}_{\geq 0}$: the genus.
($\leftrightsquigarrow q_Q$: Tits quadratic form)

For a quiver Q,

q_Q	positive definite	positive semidefinite	indefinite
Q	Dynkin	Extended Dynkin	the others
$\operatorname{mod} \Bbbk Q$	finite	tame	wild

For a smooth projective curve C,

g_C	0	1	≥ 2
C	$\mathbb{P}^1$: the projective line	E: an elliptic curve	general type
$\operatorname{coh} C$	(VB-)finite	tame	wild

Today's aim (Recall)

Classifying torsion-free classes of $\operatorname{coh} C$ when $C = \mathbb{P}^1$ or E.

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Known results for subcategories of coh X

A subcategory of an abelian category is...

- Serre subcategory :⇔ closed under sub, quot, and ext.
- torsion class (tors) :⇔ closed under quot, and ext.
- torsion-free class (torf) :⇔ closed under sub, and ext.
- wide :⇔ closed under kernels, cokernels, and ext.

$$\text{torf} \Leftarrow \text{Serre} \Rightarrow \text{tors}, \qquad \text{Serre} \Rightarrow \text{wide}$$

Known classification results for noetherian schemes

Gabriel 1962: Serre subcategories of coh X for a noetherian scheme X (especially, mod R for a commutative noetherian ring R).

Takahashi 2008, Stanley-Wang 2011, Enomoto 2023: Wide subcategories, torsion(-free) classes, and many classes of subcategories of mod R.

S. 2024: The above classifications also hold for L-closed subcategories of coh X.

Problem

Classify subcategories of coh X which is not L-closed.

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torf $\Leftarrow$ Serre $\Rightarrow$ tors

Serre $\Downarrow$ wide

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For a smooth projective curve C, the following is known.

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Krause-Stevenson 2019: Wide subcategories of coh $\mathbb{P}^1$.
($\leftrightsquigarrow$ thick subcategories of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathbb{P}^1)$)

Hirano-Ouchi 2022: Wide subcategories of coh E.
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Chen-Lin-Ruan 2023: Torsion classes of coh $\mathbb{P}^1$ and coh E.
($\leftrightsquigarrow$ torsion pair of coh $\mathbb{P}^1$ and coh E)

Today's aim (Recall)

Classifying torsion-free classes of coh C when $C = \mathbb{P}^1$ or E.

Note: {torsion pairs} $\not\leftrightarrow$ {torsion-free classes} since coh C is not artinian.

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The AR quiver of coh $\mathbb{P}^1$

- coh $\mathbb{P}^1 \simeq \operatorname{qgr} S \left(= \operatorname{mod}^{\mathbb{Z}} S / \operatorname{mod}^{\mathbb{Z}}_{\mathrm{fd}} S\right)$, where $S = \Bbbk[x, y]$.
- $\exists$ essentially surjective functor $Q : \operatorname{mod}^{\mathbb{Z}} S \to \operatorname{coh} \mathbb{P}^1$.
- Set $\mathscr{O}(n) := Q(S(n))$ $(n \in \mathbb{Z})$, and
  $\mathscr{O}_{d[a:b]} := Q\left(S/(bx - ay)^d\right)$ $(d \in \mathbb{Z}_{\geq 1}, [a : b] \in \mathbb{P}^1(\Bbbk))$.

The AR quiver of coh $\mathbb{P}^1$:

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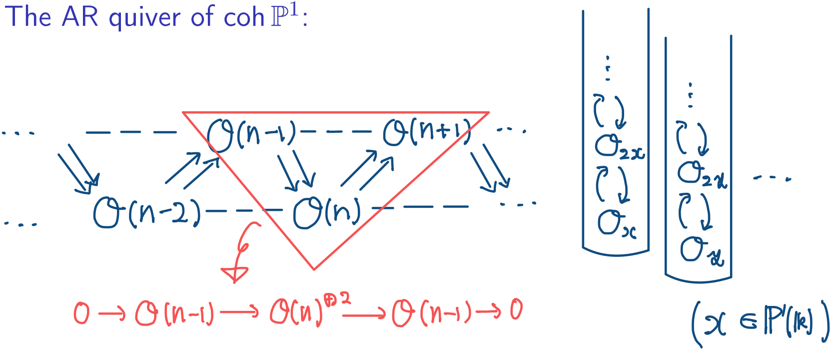
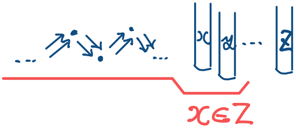


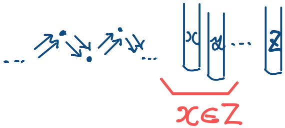
The torsion-free classes of coh $\mathbb{P}^1$:
(= sub, ext-closed)

For $Z \subseteq \mathbb{P}^1(\Bbbk)$,

• $\mathcal{X}(Z)$:

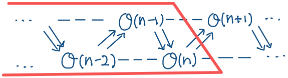


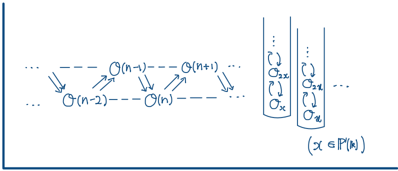
• $\mathcal{Y}(Z)$:

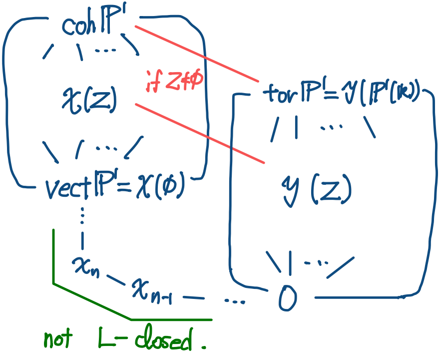


For $n \in \mathbb{Z}$,

• $\mathcal{X}_n$:







Contents

Stability conditions

C: smooth projective curve.

Definition

1. The slope of $\mathscr{F} \in \mathsf{coh}\, C$ is defined by
$$\mu(\mathscr{F}) := \frac{\deg \mathscr{F}}{\operatorname{rk} \mathscr{F}} \in \mathbb{Q} \cup \{\infty\}.$$
2. $\mathscr{F} \in \mathsf{coh}\, C$: semistable $:\Longleftrightarrow \mu(\mathscr{G}) \leq \mu(\mathscr{F})$ for $\forall$ subobject $\mathscr{G}$ of $\mathscr{F}$.
3. $\mathsf{coh}_\mu C :=$ {semistable of slope μ} $(\mu \in \mathbb{Q} \cup \{\infty\})$.

Properties

1. $\operatorname{Hom}(\mathsf{coh}_\mu C, \mathsf{coh}_\nu C) = 0$ if $\mu > \nu$.
2. $$\mathsf{coh}\, C = \bigcup_{n \geq 1} \bigcup_{\mu_1 > \mu_2 > \cdots > \mu_n} (\mathsf{coh}_{\mu_1} C) * (\mathsf{coh}_{\mu_2} C) * \cdots * (\mathsf{coh}_{\mu_n} C).$$

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1. $\operatorname{Hom}(\operatorname{coh}_\mu C, \operatorname{coh}_\nu C) = 0$ if $\mu > \nu$. $\left(\mathcal{X} * \mathcal{Y} := \{ M \mid 0 \to X \to M \to Y \to 0 \} \right)$
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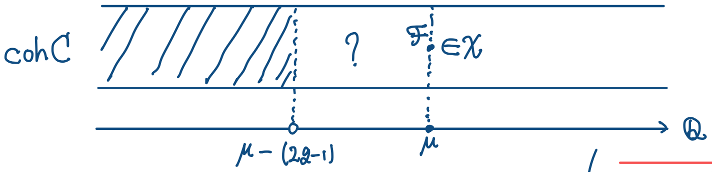
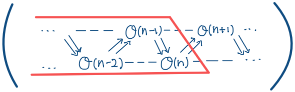
coh C | ... / $\operatorname{coh}_\nu C$ / ... / $\operatorname{coh}_\mu C$ / ... $\quad \nu < \mu$

C: smooth projective curve of genus g.

Proposition (S.)

$\mathcal{X} \subseteq \operatorname{coh} C$: torsion-free class, $\mathscr{F} \in \mathcal{X}$: semistable of slope $\mu \in \mathbb{Q}$.
Then $\mathcal{X}$ contains all the semistable sheaves of slope $< \mu - (2g-1)$.





E: an elliptic curve (i.e., $g = 1$).
We have the following by the previous proposition:

Corollary

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In this case, we obtain a better version of this corollary.

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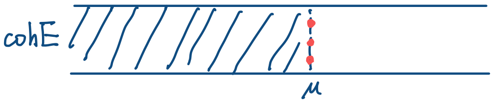
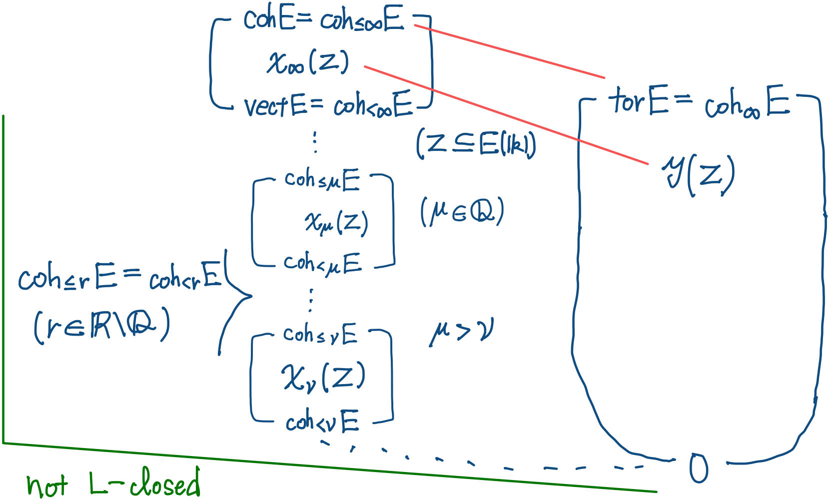


The torsion-free classes of coh E:



Thank you for your attention.