Quasi-hereditary structures and tilting modules

Mayu Tsukamoto, Takahide Adachi, Aaron Chan, Yuta Kimura

Yamaguchi University, Yamaguchi University, Nagoya University, Hiroshima Institute of Technology

17th September 2024

Introduction

Quasi-hereditary algebra [Cline–Parshall–Scott'88]

Quasi-hereditary algebra $(A, \lhd)$ consists of an algebra A and a partial order $\lhd$ on the complete set of simple A-modules satisfying certain conditions

Introduction

Quasi-hereditary algebra [Cline–Parshall–Scott'88]

Quasi-hereditary algebra $(A, \lhd)$ consists of an algebra A and a partial order $\lhd$ on the complete set of simple A-modules satisfying certain conditions

Question

- Does there exist a partial order to be a quasi-hereditary algebra?
- How to find a partial order to be a quasi-hereditary algebra?

Introduction

Quasi-hereditary algebra [Cline–Parshall–Scott'88]

Quasi-hereditary algebra $(A, \lhd)$ consists of an algebra A and a partial order $\lhd$ on the complete set of simple A-modules satisfying certain conditions

Question

- Does there exist a partial order to be a quasi-hereditary algebra?
- How to find a partial order to be a quasi-hereditary algebra?

Aim

Give an answer to Question from the viewpoint of tilting theory

Notation

Notation

- A: (basic) finite dimensional K-algebra, K: algebraically closed field
- $\{S(x) \mid x \in \Lambda\}$: complete set of isoclasses of simple A-modules
- e_x: primitive idempotent of A corresponding to $x \in \Lambda$
- $P(x)$: projective cover of $S(x)$
- For an A-module M, $[M : S(x)] := \dim_K \mathrm{Hom}_A(P(x), M)$

Standard modules

Fix a partial order $\lhd$ on Λ and let $x \in \Lambda$

Definition

A standard module $\Delta(x)$ is a maximal factor module of $P(x)$ such that

$$[\Delta(x) : S(y)] \neq 0 \Rightarrow y \unlhd x$$

Dually, costandard module $\nabla(x)$ is defined

Standard modules

Fix a partial order $\lhd$ on Λ and let $x \in \Lambda$

Definition

A standard module $\Delta(x)$ is a maximal factor module of $P(x)$ such that

$$[\Delta(x) : S(y)] \neq 0 \Rightarrow y \unlhd x$$

Dually, costandard module $\nabla(x)$ is defined

Standard modules depend on the choice of a partial order

$A = K(1 \to 2 \to 3) = \substack{1\\2\\3} \oplus \substack{2\\3} \oplus \substack{3}$

- $1 \lhd 2 \lhd 3 \Rightarrow \Delta(x) = S(x)$
- $3 \lhd 2 \lhd 1 \Rightarrow \Delta(x) = P(x)$
- $2 \lhd 1, 3 \Rightarrow \Delta(1) = \substack{1\\2}, \Delta(2) = \substack{2}, \Delta(3) = \substack{3}$

Quasi-hereditary algebras

$\Delta(\rhd x) := \{\Delta(y) \mid y \rhd x\}$

Definition

A pair $(A, \lhd)$ is called a quasi-hereditary algebra if for all $x \in \Lambda$,

- $[\Delta(x) : S(x)] = 1$
- there is an exact sequence $0 \to K(x) \to P(x) \to \Delta(x) \to 0$ such that $K(x) \in \mathcal{F}(\Delta(\rhd x))$

We call the partial order $\lhd$ a quasi-hereditary structure

Quasi-hereditary algebras

$\Delta(\rhd x) := \{\Delta(y) \mid y \rhd x\}$

Definition

A pair $(A, \lhd)$ is called a quasi-hereditary algebra if for all $x \in \Lambda$,

- $[\Delta(x) : S(x)] = 1$
- there is an exact sequence $0 \to K(x) \to P(x) \to \Delta(x) \to 0$ such that $K(x) \in \mathcal{F}(\Delta(\rhd x))$

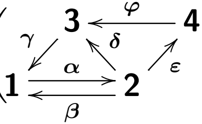
We call the partial order $\lhd$ a quasi-hereditary structure

Example

$A = K(1 \to 2 \to 3) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}$, $(A, 2 \lhd 1 \lhd 3)$: quasi-hereditary

$\Delta(1) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \Delta(2) = \begin{smallmatrix} 2 \end{smallmatrix}, \Delta(3) = \begin{smallmatrix} 3 \end{smallmatrix}$

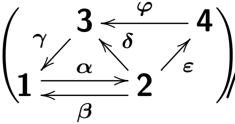
Example

$A := K\Big($ 

$\Big) / \left\langle \begin{matrix} \beta\alpha, \alpha\delta \\ \alpha\varepsilon, \gamma\alpha\beta \end{matrix} \right\rangle$

$P(1)$: $\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}$ $P(2)$: $\begin{matrix} & 2 & \\ 1 & 3 & 4 \\ & 1 & 3 \\ & 2 & 1 \\ & & 2 \end{matrix}$ $P(3)$: $\begin{matrix} 3 \\ 1 \\ 2 \end{matrix}$ $P(4)$: $\begin{matrix} 4 \\ 3 \\ 1 \\ 2 \end{matrix}$

Example

$A := K$ 

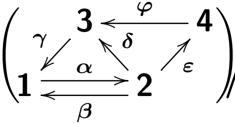
$/\!\!/ \left\langle \begin{matrix} \beta\alpha, \alpha\delta \\ \alpha\varepsilon, \gamma\alpha\beta \end{matrix} \right\rangle$

$P(1)\colon \begin{matrix} \color{red}{1} \\ 2 \\ 1 \end{matrix}$ $P(2)\colon \begin{matrix} & \color{red}{2} & \\ \color{red}{1} & 3 & 4 \\ & 1 & 3 \\ & 2 & 1 \\ & & 2 \end{matrix}$ $P(3)\colon \begin{matrix} \color{red}{3} \\ \color{red}{1} \\ \color{red}{2} \end{matrix}$ $P(4)\colon \begin{matrix} \color{red}{4} \\ \color{red}{3} \\ \color{red}{1} \\ \color{red}{2} \end{matrix}$

(1) $(A, 1 \lhd 2 \lhd 3 \lhd 4)$: quasi-hereditary

$\Delta(1)\colon \color{red}{1}$ $\Delta(2)\colon \begin{matrix} & \color{red}{2} \\ \color{red}{1} & \end{matrix}$ $\Delta(3)\colon \begin{matrix} \color{red}{3} \\ \color{red}{1} \\ \color{red}{2} \end{matrix}$ $\Delta(4)\colon \begin{matrix} \color{red}{4} \\ \color{red}{3} \\ \color{red}{1} \\ \color{red}{2} \end{matrix}$

Example

$A := K$  $\Big/\!\!\Big/ \left\langle \begin{matrix} \beta\alpha, \alpha\delta \\ \alpha\varepsilon, \gamma\alpha\beta \end{matrix} \right\rangle$

$P(1)\colon \begin{matrix} 1 \\ 2 \\ 1 \end{matrix}$ $P(2)\colon \begin{matrix} & 2 & \\ 1 & 3 & 4 \\ & 1 & 3 \\ & 2 & 1 \\ & & 2 \end{matrix}$ $P(3)\colon \begin{matrix} 3 \\ 1 \\ 2 \end{matrix}$ $P(4)\colon \begin{matrix} 4 \\ 3 \\ 1 \\ 2 \end{matrix}$

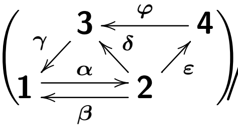
(1) $(A, 1 \lhd 2 \lhd 3 \lhd 4)$: quasi-hereditary

$\Delta(1)\colon 1$ $\Delta(2)\colon \begin{matrix} & 2 \\ 1 & \end{matrix}$ $\Delta(3)\colon \begin{matrix} 3 \\ 1 \\ 2 \end{matrix}$ $\Delta(4)\colon \begin{matrix} 4 \\ 3 \\ 1 \\ 2 \end{matrix}$

(2) $(A, 1 \lhd 2 \lhd 4 \lhd 3)$: quasi-hereditary

$\Delta(1)\colon 1$ $\Delta(2)\colon \begin{matrix} & 2 \\ 1 & \end{matrix}$ $\Delta(3)\colon \begin{matrix} 3 \\ 1 \\ 2 \end{matrix}$ $\Delta(4)\colon 4$

Example

$A := K$  $\Big/ \Big\langle \begin{array}{l} \beta\alpha, \alpha\delta \\ \alpha\varepsilon, \gamma\alpha\beta \end{array} \Big\rangle$

$P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$ $P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & 3 & 4 \\ & 1 & 3 \\ & 2 & 1 \\ & & 2 \end{smallmatrix}$ $P(3)\colon \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$ $P(4)\colon \begin{smallmatrix} 4 \\ 3 \\ 1 \\ 2 \end{smallmatrix}$

(1) $(A, 1 \lhd 2 \lhd 3 \lhd 4)$: quasi-hereditary

$\Delta(1)\colon 1$ $\Delta(2)\colon \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$ $\Delta(3)\colon \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$ $\Delta(4)\colon \begin{smallmatrix} 4 \\ 3 \\ 1 \\ 2 \end{smallmatrix}$

(2) $(A, 1 \lhd 2 \lhd 4 \lhd 3)$: quasi-hereditary

$\Delta(1)\colon 1$ $\Delta(2)\colon \begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$ $\Delta(3)\colon \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$ $\Delta(4)\colon 4$

(3) $(A, 1 \lhd 4 \lhd 2 \lhd 3)$: NOT quasi-hereditary

$\Delta(1)\colon 1$ $\Delta(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 4 \end{smallmatrix}$ $\Delta(3)\colon \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$ $\Delta(4)\colon 4$

Characteristic tilting modules

$\mathbf{\Delta} := \{\Delta(x) \mid x \in \Lambda\}, \mathbf{\nabla} := \{\nabla(x) \mid x \in \Lambda\}$

Definition-Theorem [Ringel'91]

If $(A, \lhd)$ is quasi-hereditary, then there exists a tilting module $T_{\lhd}$ such that

$$\mathrm{add}\, T_{\lhd} = \mathcal{F}(\mathbf{\Delta}) \cap \mathcal{F}(\mathbf{\nabla})$$

The module $T_{\lhd}$ is called a characteristic tilting module

Characteristic tilting modules

$\boldsymbol{\Delta} := \{\Delta(x) \mid x \in \Lambda\}, \boldsymbol{\nabla} := \{\nabla(x) \mid x \in \Lambda\}$

Definition-Theorem [Ringel'91]

If $(A, \lhd)$ is quasi-hereditary, then there exists a tilting module $T_{\lhd}$ such that

$$\operatorname{add} T_{\lhd} = \mathcal{F}(\boldsymbol{\Delta}) \cap \mathcal{F}(\boldsymbol{\nabla})$$

The module $T_{\lhd}$ is called a characteristic tilting module

Remark

- $T_{\lhd} = \bigoplus_{x \in \Lambda} T(x)$ such that $[T(x) : S(x)] = 1$ and $[T(x) : S(y)] \neq 0 \Rightarrow y \trianglelefteq x$
- [Ringel'91, Flores–Kimura–Rognerud'22]
  $\{\text{quasi-hereditary structures of } A\} \xrightarrow{\sim} \{\text{characteristic tilting } A\text{-modules}\}$

IS-tilting modules

Definition

A basic tilting module T is called an IS-tilting module if there exists a partial order $\lhd$ on Λ satisfying the following conditions:

$$\exists \text{ indec. decomp. } T = \bigoplus_{x \in \Lambda} T(x) \text{ such that } T(x)\varepsilon_x \cong S(x)\varepsilon_x,$$

where $\Lambda_x := \{y \in \Lambda \mid y \not\lhd x\}$ and $\varepsilon_x := \sum_{y \in \Lambda_x} e_y$
In this case, we write $(T, \lhd)$ to specify the associated partial order

IS-tilting modules

Definition

A basic tilting module T is called an IS-tilting module if there exists a partial order $\lhd$ on Λ satisfying the following conditions:

$$\exists \text{ indec. decomp. } T = \bigoplus_{x \in \Lambda} T(x) \text{ such that } T(x)\varepsilon_x \cong S(x)\varepsilon_x,$$

where $\Lambda_x := \{y \in \Lambda \mid y \not\lhd x\}$ and $\varepsilon_x := \sum_{y \in \Lambda_x} e_y$
In this case, we write $(T, \lhd)$ to specify the associated partial order

Remark

For a partial order $\lhd$ on Λ, $(T, \lhd)$: IS-tilting module $\Leftrightarrow$ $\exists$ indec. decomp. $T = \bigoplus_{x \in \Lambda} T(x)$ such that $[T(x) : S(x)] = 1$ and $[T(x) : S(y)] \neq 0 \Rightarrow y \unlhd x$

IS-tilting modules

Definition

A basic tilting module T is called an IS-tilting module if there exists a partial order $\lhd$ on Λ satisfying the following conditions:

$$\exists \text{ indec. decomp. } T = \bigoplus_{x \in \Lambda} T(x) \text{ such that } T(x)\varepsilon_x \cong S(x)\varepsilon_x,$$

where $\Lambda_x := \{y \in \Lambda \mid y \not\ntriangleleft x\}$ and $\varepsilon_x := \sum_{y \in \Lambda_x} e_y$

In this case, we write $(T, \lhd)$ to specify the associated partial order

Remark

For a partial order $\lhd$ on Λ, $(T, \lhd)$: IS-tilting module $\Leftrightarrow$ $\exists$ indec. decomp. $T = \bigoplus_{x \in \Lambda} T(x)$ such that $[T(x) : S(x)] = 1$ and $[T(x) : S(y)] \neq 0 \Rightarrow y \unlhd x$

Example

Characteristic tilting modules are IS-tilting

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{l} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} 3 & & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c}\alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta\end{array}\right\rangle \quad P(1)\colon \begin{smallmatrix}1\\2\\3\end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix}3\\4\\5\end{smallmatrix} \quad P(4)\colon \begin{smallmatrix}4\\5\end{smallmatrix} \quad P(5)\colon \begin{smallmatrix}5\\3\end{smallmatrix}$$

$$T := {\color{red}2} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2 & & 5\\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\\3\end{smallmatrix} \qquad {\color{red}2} \lhd$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1){:}\ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2){:}\ \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3){:}\ \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4){:}\ \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5){:}\ \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := {\color{red}2} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

$${\color{red}2} \lhd$$

$$T(1 - {\color{red}e_2}) = 3 \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$$

$$T(2){:}\ 2$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix} \qquad\qquad 2 \lhd {\color{red}3} \lhd$$

$$T(1-e_2) = {\color{red}3} \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \qquad\qquad T(2)\colon \begin{smallmatrix} 2 \end{smallmatrix}$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

$$2 \lhd {\color{red}3} \lhd$$

$$T(1-e_2) = {\color{red}3} \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$$

$$T(2)\colon 2 \qquad T(3)\colon \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$$

$$T(1-e_2-{\color{red}e_3}) = 5 \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 1$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \nwarrow^{\varphi} & & \swarrow_{\varepsilon} \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1){:}\ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2){:}\ \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3){:}\ \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4){:}\ \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5){:}\ \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix} \qquad 2 \lhd 3 \lhd {\color{red}\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}} \lhd$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \qquad T(2){:}\ 2 \quad T(3){:}\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$$

$$T(1-e_2-e_3) = {\color{red}5} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus {\color{red}1}$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ {\scriptstyle\alpha}\uparrow\downarrow{\scriptstyle\beta} & & {\scriptstyle\varphi}\nwarrow & & \swarrow{\scriptstyle\varepsilon} \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle$$

$P(1)\colon \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ $P(2)\colon \begin{smallmatrix}&2&\\1&&3\\2&&\\3&&\end{smallmatrix}$ $P(3)\colon \begin{smallmatrix}3\\4\\5\end{smallmatrix}$ $P(4)\colon \begin{smallmatrix}4\\5\end{smallmatrix}$ $P(5)\colon \begin{smallmatrix}5\\3\end{smallmatrix}$

$$T := 2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2&&5\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\\3\end{smallmatrix}$$

$$2 \lhd 3 \lhd \begin{smallmatrix}\color{red}{1}\\ \\ \color{red}{5}\end{smallmatrix} \lhd$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix}5\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$$

$T(1)\colon \begin{smallmatrix}2\\1\\2\\3\end{smallmatrix}$ $T(2)\colon 2$ $T(3)\colon \begin{smallmatrix}2\\3\end{smallmatrix}$ $T(5)\colon \begin{smallmatrix}2&&5\\&3&\end{smallmatrix}$

$$T(1-e_2-e_3) = \color{red}{5} \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \oplus \color{red}{1}$$

$$T(1-\color{red}{e_1}-e_2-e_3-\color{red}{e_5}) = 4$$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \nwarrow\varphi & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle$$

$P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$ $P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix}$ $P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$ $P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

$$2 \lhd 3 \lhd \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \lhd \color{red}{4}$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$$

$T(1)\colon \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$ $T(2)\colon 2$ $T(3)\colon \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ $T(4)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $T(5)\colon \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix}$

$$T(1-e_2-e_3) = 5 \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 1$$

$$T(1-e_1-e_2-e_3-e_5) = 4$$

Example

$$A := K\left(\begin{matrix} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \nwarrow \varphi & & \swarrow \varepsilon \\ 1 & & & 5 & \end{matrix}\right) \Big/ \left\langle \begin{matrix} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{matrix} \right\rangle$$

$P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$ $P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix}$ $P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$ $P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

$$\left(T, 2 \lhd 3 \lhd \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \lhd 4\right)\colon \text{ IS-tilting}$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix} & 5 \\ 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$$

$T(1)\colon \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$ $T(2)\colon 2$ $T(3)\colon \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ $T(4)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $T(5)\colon \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix}$

$$T(1-e_2-e_3) = 5 \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 1$$

$$T(1-e_1-e_2-e_3-e_5) = 4$$

Example

$$A := K\left(\begin{array}{ccc} 2 \xrightarrow{\gamma} 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & \nwarrow^{\varphi} \quad {}^{\varepsilon}\swarrow & \\ 1 & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle$$

$P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$ $P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix}$ $P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$ $P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$$

$$\left(T, 2 \lhd 3 \lhd \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \lhd 4\right)\colon \text{ IS-tilting}$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix} 5 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}$$

$T(1)\colon \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix}$ $T(2)\colon 2$ $T(3)\colon \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ $T(4)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}$ $T(5)\colon \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix}$

$$T(1-e_2-e_3) = 5 \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 1$$

$$T(1-e_1-e_2-e_3-e_5) = 4$$

condition: $T(x)\varepsilon_x \cong S(x)\varepsilon_x$,

where $\varepsilon_x := \sum_{y \in \{y \in \Lambda \mid y \ntrianglelefteq x\}} e_y$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \nwarrow\varphi & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c}\alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta\end{array} \right\rangle$$

$P(1){:}\ \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ $P(2){:}\ \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix}$ $P(3){:}\ \begin{smallmatrix}3\\4\\5\end{smallmatrix}$ $P(4){:}\ \begin{smallmatrix}4\\5\\3\end{smallmatrix}$ $P(5){:}\ \begin{smallmatrix}5\\3\end{smallmatrix}$

$$T := 2 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2 & & 5\\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\\3\end{smallmatrix}$$

$$\left(T, 2 \lhd 3 \lhd \begin{smallmatrix}\color{red}{1}\\ \\ \color{red}{5}\end{smallmatrix} \lhd \color{red}{4}\right){:}\ \textbf{IS-tilting}$$

$$T(1-e_2) = 3 \oplus \begin{smallmatrix}5\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$$

$T(1){:}\ \begin{smallmatrix}2\\1\\2\\3\end{smallmatrix}$ $T(2){:}\ 2$ $T(3){:}\ \begin{smallmatrix}2\\3\end{smallmatrix}$ $T(4){:}\ \begin{smallmatrix}3\\4\\5\end{smallmatrix}$ $T(5){:}\ \begin{smallmatrix}2 & & 5\\ & 3 & \end{smallmatrix}$

$$T(1-e_2-e_3) = 5 \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \oplus 1$$

$$T(1-e_1-e_2-e_3-e_5) = 4$$

condition: $T(x)\varepsilon_x \cong S(x)\varepsilon_x$,

where $\varepsilon_x := \sum_{y \in \{y \in \Lambda \mid y \ntrianglelefteq x\}} e_y$

Main results

Theorem 1 [Adachi–Chan–Kimura–T]

If there exists an IS-tilting A-module $(T, \lhd)$, then $(A, \lhd)$ is quasi-hereditary

Main results

Theorem 1 [Adachi–Chan–Kimura–T]

If there exists an IS-tilting A-module $(T, \lhd)$, then $(A, \lhd)$ is quasi-hereditary

Strategy

- $(T, \lhd)$: IS-tilting A-module and $B := \mathrm{End}_A(T)$
  $\Rightarrow$ ① $({}_B T, \lhd^{\mathrm{op}})$: IS-tilting B^{op}-module ② $(B, \lhd^{\mathrm{op}})$: quasi-hereditary
- $(T, \lhd)$: IS-tilting A-module $\Rightarrow$ $({}_B T, \lhd^{\mathrm{op}})$: IS-tilting B^{op}-module ($\because$ ①)
  $\Rightarrow$ $(\mathrm{End}_{B^{\mathrm{op}}}(T), \lhd)$: quasi-hereditary ($\because$ ②)
  $\Rightarrow$ $(A, \lhd)$: quasi-hereditary ($\because$ $\mathrm{End}_{B^{\mathrm{op}}}(T) \cong A$)

Main results

Theorem 2 [ACKT]

There exist mutually inverse bijections

$$\{\text{quasi-hereditary structures of } A\} \rightleftarrows \{\text{IS-tilting } A\text{-modules}\}$$

$$\lhd \mapsto T_{\lhd}$$

$$\lhd \leftarrow\!\shortmid T \quad ((T, \lhd)\text{: IS-tilting})$$

In particular, characteristic tilting modules coincide with IS-tilting modules

Main results

Theorem 2 [ACKT]

There exist mutually inverse bijections

$$\{\text{quasi-hereditary structures of } A\} \rightleftarrows \{\text{IS-tilting } A\text{-modules}\}$$

$$\lhd \mapsto T_{\lhd}$$

$$\lhd \leftarrow\!\shortmid\; T \quad ((T, \lhd)\text{: IS-tilting})$$

In particular, characteristic tilting modules coincide with IS-tilting modules

Remark

If $(T, \lhd)$ and $(T, \lhd')$ are IS-tilting modules, then $\Delta(x) = \Delta'(x)$ for all $x \in \Lambda$

Example

$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{array}{c} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{array} \right\rangle \quad P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix} \qquad 2 \lhd 3 \lhd \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \lhd 4$$

$(T, \lhd)$: **IS-tilting module**

Example

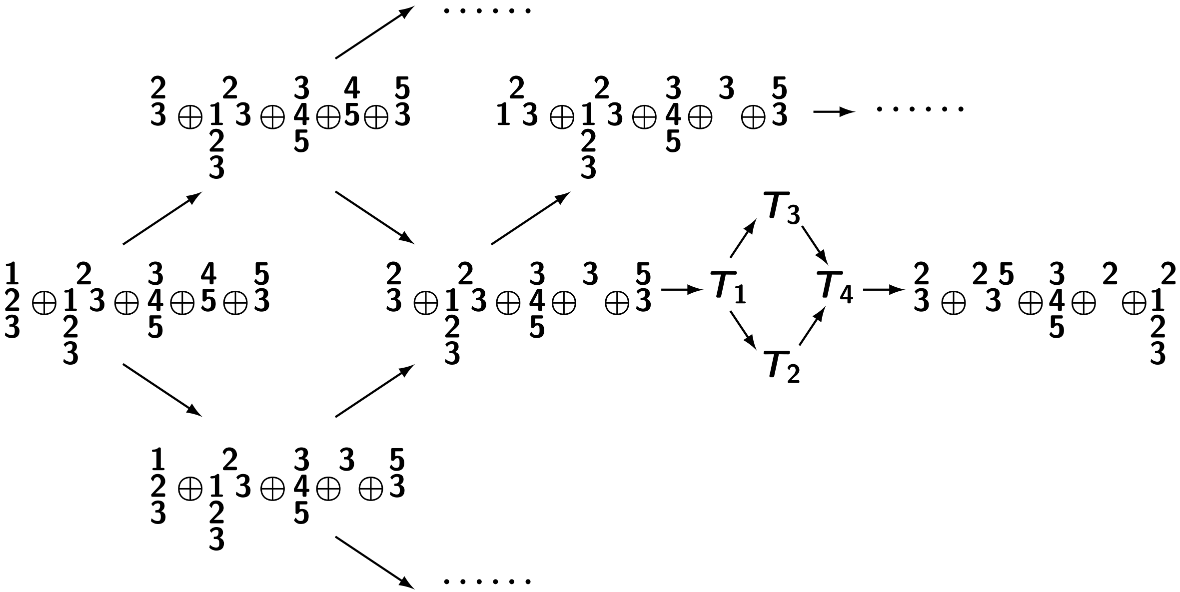
$$A := K\left(\begin{array}{ccccc} 2 & \xrightarrow{\gamma} & 3 & \xrightarrow{\quad\delta\quad} & 4 \\ \alpha\uparrow\downarrow\beta & & \varphi\nwarrow & & \swarrow\varepsilon \\ 1 & & & 5 & \end{array}\right) \Big/ \left\langle \begin{matrix} \alpha\beta, \gamma\delta \\ \varepsilon\varphi, \varphi\delta \end{matrix} \right\rangle \quad P(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \quad P(2)\colon \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ 2 & & \\ 3 & & \end{smallmatrix} \quad P(3)\colon \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \quad P(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \quad P(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$

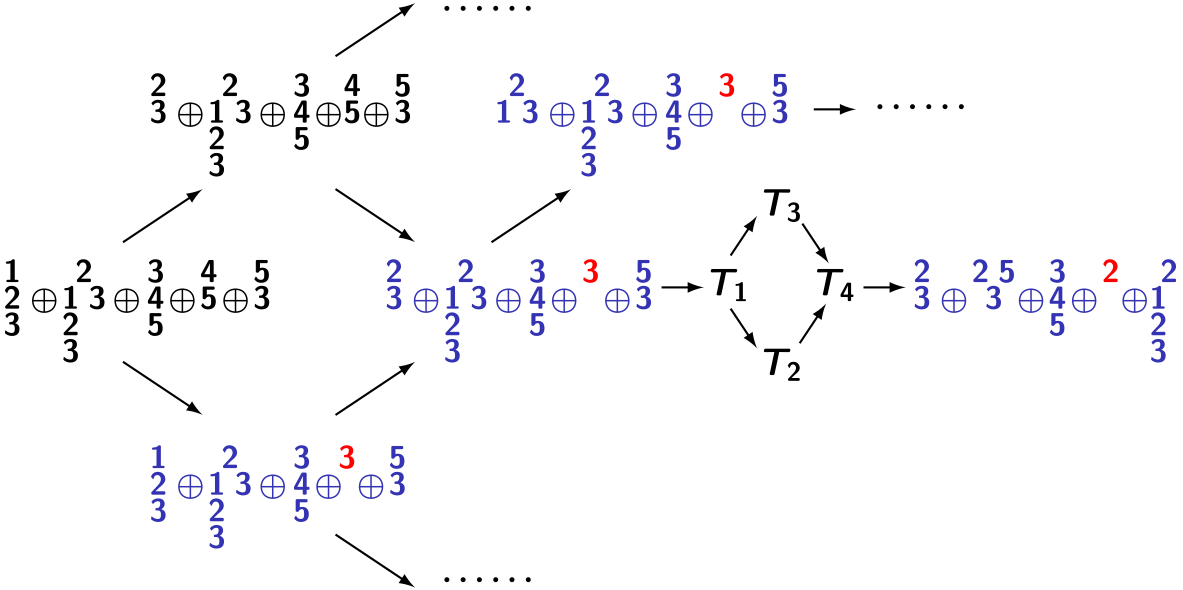
$$T := 2 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 5 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 3 \end{smallmatrix} \qquad 2 \lhd 3 \lhd \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \lhd 4$$

$(T, \lhd)$: **IS-tilting module**

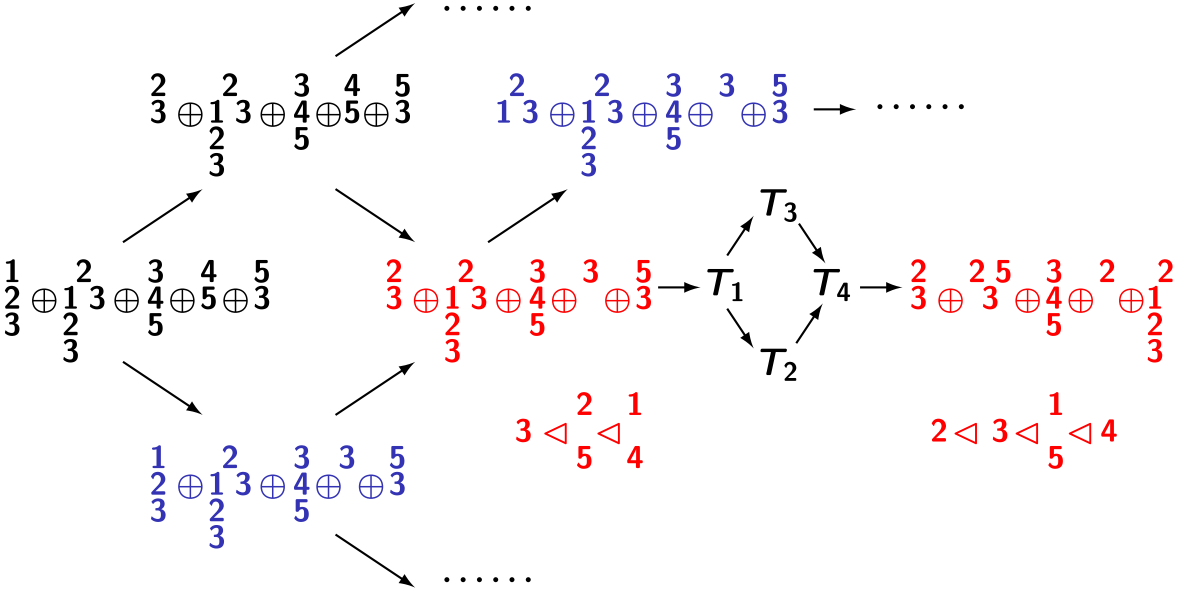
$\therefore (A, \lhd)$: **quasi-hereditary**

$$\Delta(1)\colon \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \qquad \Delta(2)\colon \ 2 \qquad \Delta(3)\colon 3 \qquad \Delta(4)\colon \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \qquad \Delta(5)\colon \begin{smallmatrix} 5 \\ 3 \end{smallmatrix}$$


T1
T2
T3
T4



T_1
T_2
T_3
T_4



Main results

Question

When are all tilting modules IS-tilting?

Main results

Question

When are all tilting modules IS-tilting?

Theorem 3 [ACKT]

For an algebra A, the following statements are equivalent

- All tilting A-modules are IS-tilting
- A: quadratic linear Nakayama algebra
  ($:\Leftrightarrow A \cong K(1 \to 2 \to \cdots \to n)/\langle \text{quadratic relations} \rangle$)

Main results

Question

When are all tilting modules IS-tilting?

Theorem 3 [ACKT]

For an algebra A, the following statements are equivalent

- All tilting A-modules are IS-tilting
- A: quadratic linear Nakayama algebra
  (:$\Leftrightarrow A \cong K(1 \to 2 \to \cdots \to n)/\langle$quadratic relations$\rangle$)

Remark

We can give recursive formulas for enumerating tilting modules (in Theorem 3)