Connections on dg modules and naïve liftings

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Background

Let $R \to S(= R/I)$ be commutative noetherian rings.

- An S-module M is said to be liftable to R if there exists an R-module N such that

$$N \otimes_R S \cong M \quad \text{and} \quad \mathrm{Tor}_i^R(N, S) = 0 \quad \text{for} \quad i > 0.$$

- Equivalently, there is an acyclic free complex F over R such that $F \otimes_R S$ gives an S-free resolution of M.

In the case $I = (x)$ where x is a non-zero divisor in R, Auslander-Ding-Solberg (1993) show:

- there is an obstruction class e_M for liftability in $\mathrm{Ext}_S^2(M, M)$.
- since $e_{M_1 \oplus M_2} = e_{M_1} + e_{M_2}$, the module class of $e_M = 0$ is closed under direct summand, however a direct summand of a liftable module is not necessarily liftable.
- An S-module M is said to be weakly liftable to R if $M \oplus M'$ is liftable for some M'. This is equivalent to $e_M = 0$.

Example

Ler $R = k[x,y]/(x^2 - y^3) \twoheadrightarrow S = R/(y) \cong k[x]/(x^2)$.

$$\cdots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x & y \\ y^2 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & -y \\ -y^2 & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \\ y^2 & x \end{pmatrix}} R^2 \to M \to 0,$$

is an R-free resolution, and taking modulo (y) we have

$$\cdots \longrightarrow S^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}} S^2 \to k \oplus k \to 0.$$

Thus S-module $k \oplus k$ is liftable to R. However one can see that k is not liftable, since

$$\cdots \longrightarrow S \xrightarrow{x} S \xrightarrow{x} S \longrightarrow k \to 0$$

cannot be lifted. This k is weakly liftable.

Expand into the dg world

R : a commutative ring

- (A, d^A) : a commutative dg R-algebra （可換次数付き微分代数）
  1. $A = \bigoplus_{i \geqq 0} A_i$: a non-negatively graded R-algebra, commutative in the following sense;

     $$ab = (-1)^{|a||b|} ba, \quad \text{and} \quad a^2 = 0 \ \text{ if } |a| \text{ is odd}$$

  2. $d^A : A \to A(-1)$ is graded R-linear with $(d^A)^2 = 0$.
  3. (Leibniz rule)

     $$d^A(ab) = d^A(a)b + (-1)^{|a|} a d^A(b)$$

     for $a, b \in A$.

Expand into the dg world

A: a commutative dg R-algebra

- $N = (N, \partial^N)$: a dg A-module （次数付き微分 A-加群）
  1. $N = \bigoplus_{i \in \mathbb{Z}} N_i$: a graded (right) A-module
  2. $\partial^N : N \to N(-1)$ is graded R-linear with $(\partial^N)^2 = 0$.
  3. (Leibniz rule)

$$\partial^N(xa) = \partial^N(x)a + (-1)^{|x|} x d^A(a)$$

  for $x \in N, a \in A$.

- N is a semi-free dg A-module if N is a graded free A-module with well-ordered free basis E with the following property:

$$\partial^N(e) \in \{e' \in E \mid e' < e\}A \quad \text{for} \quad e \in E.$$

  E is called a semi-free basis of N.

Naïve lifting

$A \to B$: a dg R-algebra homomorphism of commutative dg R-algebras

Definition

We say $Y \in D(B)$ is naïvely liftable to A if $\pi_Y : Y|_A \otimes^{\mathbf{L}}_A B \to Y$ has a right inverse in $D(B)$, i.e. $\exists \rho : Y \to Y|_A \otimes^{\mathbf{L}}_A B$ in $D(B)$ such that $\pi_Y \circ \rho = 1_Y$.

N : a semi-free DG B-module with semi-free basis $E = \{e_\lambda\}_{\lambda \in \Lambda}$.

- N is liftable to $A \iff N \cong M \otimes_A B$ as dg B-modules for $\exists$ dg A-module M.
- N is naïvely liftable to A
  $\iff$ the natural dg B-module homomorphism
  $\pi_N : N|_A \otimes_A B \to N$ $(n \otimes b \mapsto nb)$ is a split epimorphism,
  i.e. $\exists$ dg B-homomorphism $\rho_N : N \to N|_A \otimes_A B$ s.t., $\pi_N \circ \rho_N = \mathrm{id}_N$.
- liftable $\Rightarrow$ naïve liftable.

Naïve lifting conjecture

Conjecture (Naive lifting conjecture)

Under the setup $A \to B$, let N be a (semi-free, non-negatively graded) dg B-module. Assume the AR assumption, i.e.

1. B is a direct summand of N,
2. $N|_A$ is perfect,
3. $\mathrm{Hom}_{D(B)}(N, N(i)) = 0$ for $i \neq 0$.

Then, N is naively liftable to A, and hence N is perfect over B.

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N is perfect over $B \Leftrightarrow \exists$ a dg B-module homomorphism $L \to N$ that is quasi-isomorphic and L is finite free as a graded B-module.

Obstruction class for naïve lifting

Setting

$A \to B$: commutative dg R-algebras
N : a semi-free dg B-mdoule with semi-free basis $E = \{e_\lambda\}_{\lambda \in \Lambda}$.

$\pi_B : B^e = B^o \otimes_A B \to B \,(a \otimes b \mapsto ab)$: the multiplication
$J = \mathrm{Ker}(\pi_B)$: the diagonal ideal
$\exists$ a short exact sequence of dg B^e-modules:

$$0 \to J \to B^e \xrightarrow{\pi_B} B \to 0.$$

$\rightsquigarrow$ applying $N \otimes_B -$, we have an exact sequence of dg B-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_B J & \longrightarrow & N \otimes_B B^e & \xrightarrow{\mathrm{id}_N \otimes \pi_B} & N \otimes_B B & \longrightarrow & 0. \\ & & & & \| & & \| & & \\ & & & & N|_A \otimes_A B & \xrightarrow{\pi_N} & N & & \end{array}$$

Therefore, the obstruction class for naive lifting of N lies in $\mathrm{Ext}^1_B(N, N \otimes_B J) = \mathrm{Hom}_{K(B)}(N, N \otimes_B J(-1))$, where $K(B)$ is the homotopy category of dg B-modules.

Simple free extensions

Definition

A semi-free dg B-module N is called **weakly liftable** to A if a finite direct sum $N \oplus N(-a_1) \oplus \cdots \oplus N(-a_r)$ where $a_i > 0$ is liftable to A.

Let $A \to B = A\langle X \rangle$ be a free extension with a single variable X, and let $N = (N, \partial^N)$ be a semi-free dg B-module that is bounded below.

Theorem (Nasseh-Ono-Yoshino, JAlg 2022)

1. *There is a well-defined obstruction class* $[j_X(\partial^N)] \in \operatorname{Ext}_B^{|X|+1}(N, N)$, *where* j_X *is the j-operator.*
2. *If* $|X|$ *is even, then* $[j_X(\partial)] = 0 \iff (N, \partial)$ *is liftable.*
3. *If* $|X|$ *is odd, then* $[j_X(\partial)] = 0$
   $\iff (N, \partial) \oplus (N(-|X|), -\partial)$ *is liftable*
   $\iff (N, \partial)$ *is weakly liftable.*

Generalized Atiyah class

- $E = \{e_\lambda\}_{\lambda \in \Lambda}$: a semi-free basis of N
- $\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda}$ for $e_\lambda \in E$
- Define a graded B-linear mapping $\omega_N : N \to N \otimes_B J(-1)$ by

$$\omega_N(e_\lambda) = \sum_{\mu<\lambda} e_\mu \otimes \delta(b_{\mu\lambda})$$

  where $\delta(b) = b \otimes 1 - 1 \otimes b \in J$ for $b \in B$.
- ω_N is a dg B-module homomorphism, and it defines the homotopy class $[\omega_N]$ in $\mathrm{Ext}^1_B(N, N \otimes_B J) = \mathrm{Hom}_{K(B)}(N, N \otimes_B J(-1))$.
- We call $[\omega_N]$ the generalized Atiyah class of N.

Theorems

Theorem (Nasseh-Ono-Yoshino, JCommAlg 2024)

❶ N is naively liftable to A.

❷ $[\omega_N] = 0$ in $\mathrm{Ext}^1_B(N, N \otimes_B J)$.

Theorem (Nasseh-Ono-Yoshino, MathZ 2022)

Let B be a dg smooth dg algebra over A (i.e. the Kähler differential $\Omega_A(B) := J/J^2$, moreover all $J^{[m]}/J^{[m+1]}$ are finite semi-free over B), and let N be a semi-free dg B-module that is bounded below. Then,

$$\begin{aligned} \mathrm{Ext}^i_B(N, N) = 0 \ (i > 0) \quad &\Rightarrow \quad \mathrm{Ext}^i_B(N, N \otimes_B J) = 0 \ (i > 0) \\ &\Rightarrow \quad N \textit{ is naïvely liftable to } A. \end{aligned}$$

Diagonal tensor algebra

Let $A \to B$ be as in the initial setting (B is semi-free over A).

Definition

$T = T_B(J(-1)) = \bigoplus_{i \geq 0} J^{\otimes i}(-i) = B \oplus J(-1) \oplus J(-1)^{\otimes 2} \oplus \cdots$

where $J(-1)^{\otimes i} = (J \otimes_B J \otimes_B \cdots \otimes_B J)(-i)$ (i-times).

1. T is a $\mathbb{Z}^2$-graded algebra ; $T_n^i = J_{n-i}^{\otimes i}$ for $(n, i) \in \mathbb{Z}^2$. Call i the tensor-degree, while n is the dg-degree.
2. T is a graded dg R-algebra. (graded in tensor-degree)
3. $B \otimes_A T$ has a structure of semi-free dg B^e-module that resolves B. (Equivalent to the reduced bar resolution by Cuntz-Quillen (1995))

A tensor-graded dg T-module M is a direct sum $M = \bigoplus_{i\in\mathbb{Z}} M^i$ of dg B-modules with a bunch of dg B-module homomorphisms $a_M^i : M^i \otimes_B J(-1) \to M^{i+1}$ for $i \in \mathbb{Z}$. A tensor-graded dg T-module homomorphism $f : M = \bigoplus_{i\in\mathbb{Z}} M^i \to N = \bigoplus_{i\in\mathbb{Z}} N^i$ is a collection of dg B-module homomorphisms $\{f^i\}_{i\in\mathbb{Z}}$ which make the following squares commutative in $C(B)$:

$$\begin{array}{ccc} M^i \otimes_B J(-1) & \xrightarrow{a_M^i} & M^{i+1} \\ \Big\downarrow{\scriptstyle f_i\otimes 1_{J(-1)}} & & \Big\downarrow{\scriptstyle f^{i+1}} \\ N^i \otimes_B J(-1) & \xrightarrow[a_N^i]{} & N^{i+1} \end{array}$$

We denote by $C^{gr}(T)$ the category of all tensor-graded dg T-modules and tensor-graded dg T-homomorphisms. Obviously $C^{gr}(T)$ is an abelian category.

abelian category $C^{gr}(T) \Rightarrow$ homotopy category $K^{gr}(T)$
$\Rightarrow$ derived category $D^{gr}(T)$

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Two grading for $X \in C^{gr}(T)$

1. $X(n)$ the nth shift in dg grading; $X(n)_m = X_{n+m}$, $\partial^{X(n)} = (-1)^n \partial^X$
2. $X[n]$ the nth shift in tensor-grading; If $X = \bigoplus_i X^i$ with $X^i J(-1) \subset X^{i+1}$, then $X[n] = \bigoplus_i Y^i$ where $Y^i = X^{i+n}$.

For $X, Y \in D^{gr}(T)$

$$^*\mathrm{Hom}_{D^{gr}(T)}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{D^{gr}(T)}(X, Y[i]).$$

$$\Gamma_X := {}^*\mathrm{End}_{D^{gr}(T)}(X) = {}^*\mathrm{Hom}_{D^{gr}(T)}(X,X).$$

Γ_X is a graded ring in which the multiplication is just a composition of morphisms, and its degree i part Γ^i is $\mathrm{Hom}_{D^{gr}(T)}(X, X[i])$:

Let $A \to B$ be as in the initial setup and N a semi-free dg B-module.

Set

$\Lambda := \mathrm{End}_{D(B)}(N), \quad \Gamma := \Gamma_{N\otimes_B T} = {}^*\mathrm{End}_{D^{gr}(T)}(N \otimes_B T).$

There is a natural ring homomorphisms

$$\Lambda \underset{p}{\overset{i}{\rightleftarrows}} \Gamma$$

$i(f) = f \otimes_B T$ for $f \in \Lambda$, and $p(\varphi) = \varphi \otimes_T B$ for $\varphi \in \Gamma$.

Since $p \circ i = id_\Lambda$, Λ is a subring of Γ with retraction.

Obstruction

Definition

Let N be a sf dg B-module with sf basis $\{e_\lambda\}$ which satisfies

$$\partial^N(e_\lambda) = \sum_{\mu<\lambda} e_\mu b_{\mu\lambda} \quad (b_{\mu\lambda} \in B)$$

Define the morphism $\omega = \omega_N : N \otimes_B T \to N \otimes_B T[1]$ in $C^{gr}(T)$; by

$$\omega(e_\lambda \otimes t) = \sum_{\mu<\lambda} e_\mu \otimes_B \delta(b_{\mu\lambda}) \otimes_B t,$$

where $t \in T$.

$$\omega \in \Gamma^1 = \mathrm{Hom}_{D^{gr}(T)}(N \otimes_B T, N \otimes_B T[1]).$$

Remark

1. The definition of ω is independent of the choice of semi-free basis.
2. N is naively liftable to $A \Leftrightarrow \omega = 0$ in $D^{gr}(T)$.

Theorems [Nasseh-Ono-Yoshino, JPAA(2024)]

Assumption (AR-1)

Adding to the initial setting for $A \to B$, we assume the following:

1. N is a sf dg B-module that is non-negatively graded.
2. $B|_A$ and $N|_A$ are perfect dg A-modules.
3. $\mathrm{Hom}_{D(B)}(N, B(-n)) = 0$ whenever $n \neq 0$.

Theorem (1)

Under the assumption (AR-1), we have $\Gamma = \Lambda[\ \omega\ ]$, i.e. every element of Γ is of the polynomial form $\alpha_0 + \omega\alpha_1 + \omega^2\alpha_2 + \cdots + \omega^n\alpha_n$ with $\alpha_i \in \Lambda$.

Furthermore, the multiplication by ω induces bijections $\Gamma^i \xrightarrow{\omega\circ} \Gamma^{i+1}$ for all $i > 0$.

Theorem (2)

Under the assumption (AR-1), there is an exact sequence of graded right Γ-modules

$$0 \longrightarrow P \longrightarrow \Gamma \xrightarrow{\omega\circ} \Gamma \longrightarrow \Lambda \longrightarrow 0\ ,$$

where P is a graded Γ-ideal concentrated in degree 0 (hence a Λ-ideal) and

$$P = \{f \in \Lambda \mid f \text{ factors through a direct sum of copies of } B\}.$$

Theorem (3)

Under the assumption (AR-1), the following conditions are equivalent for N.

1. N *is naively liftable to* A.
2. $\omega = 0$ *as an element of* Γ.
3. ω *is nilpotent in* Γ, *i.e.* $\omega^n = 0$ *for some* $n > 0$.
4. *The natural mapping* $\Lambda \to \Gamma$ *is an isomorphism.*
5. Γ *is finitely generated as a (right) module over* Λ.
6. $\Gamma^i = 0$ *for all* $i > 0$.
7. $\Gamma^i = 0$ *for some* $i > 0$.
8. $\Gamma^1 \cong \mathrm{Hom}_{D(B)}(N, N \otimes_B J(-1)) = 0$

Theorem (4)

Assume the same assumption as in the Naive Lifting Conjecture, i.e.

1. *B is a direct summand of N,*
2. *$N|_A$ is perfect,*
3. $\mathrm{Hom}_{D(B)}(N, N(i)) = 0$ *for* $i \neq 0$.

Then the following hold.

- (i) ${}^*\mathrm{Hom}_{D^{gr}(T)}(N \otimes_B T, N \otimes_B T(i)) = 0$ *for* $i > 0$.
- (ii) Λ *has finite projective dimension as right* Γ*-module.*

Derivation

Let $A \to B$ be dg R-algebras, and let L be a dg B^e-module.

- A graded A-linear mapping $D : B \to L(n)$ of degree $n \in \mathbb{Z}$ is called an A-derivation if the Leibniz rule holds;

$$D(b_1 b_2) = D(b_1)b_2 + (-1)^{|b_1|n} b_1 D(b_2), \quad \text{for} \;\; b_1, b_2 \in B.$$

- $\mathrm{Der}_A(B, L)_n$ is the set of all A-derivations of degree n.
- Set ${}^*\mathrm{Der}_A(B, L) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}_A(B, L)_n$, which is a graded B-module.
- Define the differential $\partial^{{}^*\mathrm{Der}}$ on ${}^*\mathrm{Der}_A(B, L)$ by

$$\partial^{{}^*\mathrm{Der}}(D) = \partial^L \circ D - (-1)^{|D|} D \circ d^B \;\; \text{for} \;\; D \in {}^*\mathrm{Der}_A(B, L).$$

- $({}^*\mathrm{Der}_A(B, L), \partial^{{}^*\mathrm{Der}})$ is a dg B-module.
- It is known that there is an isomorphism of dg B-modules;

$${}^*\mathrm{Der}_A(B, L) \cong {}^*\mathrm{Hom}_{B^e}(J, L) \;\; \Big(:= \bigoplus_n \mathrm{Hom}_{grB^e-mod}(J, L(n))\Big)$$

Derivation2

Remark that

- In the case $L = B$, ${}^*\mathrm{Der}_A(B) := {}^*\mathrm{Der}_A(B, B)$ has a Lie algebra structure by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1$$

  .

- In the case $R = A$, the differential d^B is in $\mathrm{Der}_R(B)_{-1}$.
- If either $B = A\langle X_1, X_2, \ldots\rangle$ or $B = A[X_1, X_2, \ldots]$, the partial derivative $\partial/\partial X_n$ is an A-derivation of degree $-|X_n|$.

Connection

Recall the definition of dg B-module N:

$$\partial^N(xb) = \partial^N(x)b + (-1)^{|x|}xd^B(b) \quad \text{for} \quad x \in N, b \in B.$$

- Let $A \to B$ be dg R-algebras and N be a dg B-module.
- Let D be an A-derivation on B, i.e. $D \in \mathrm{Der}_A(B)_n$. A graded A-linear mapping $\psi : N \to N(m)$ is called a D-connection on N if $|D| = m$ and it satisfies

$$\psi(xb) = \psi(x)b + (-1)^{m|x|}xD(b) \quad \text{for} \quad x \in N, b \in B.$$

- Denote by $\mathrm{Conn}(N)_m$ the set of all connections of degree m (i.e. all D-connections where D runs through all A-derivations of degree m).
- Set

$$^*\mathrm{Conn}(N) = \bigoplus_m \mathrm{Conn}(N)_m$$

Remark that ∂^N is a d^B-connection.
Also remark that 0-connections = B-linear mappings.

Connection2

Lemma

1. If ψ_i is a D_i-connection $(i = 1, 2)$ with $|D_1| = |D_2|$, then $\psi_1 + \psi_2$ is a $D_1 + D_2$-connection.
2. If ψ is a D-connection and $b \in B$ homogeneous, then $\psi \cdot b$ is a $D \cdot b$-connection.
3. If ψ_i is a D_i-connection $(i = 1, 2)$, $[\psi_1, \psi_2]$ is a $[D_1, D_2]$-connection.
4. Set ${}^*\mathrm{Conn}(N) = \bigoplus_n \mathrm{Conn}(N)_n$ and define the differential by

$$\partial^{{}^*\mathrm{Conn}}(\psi) = [\partial^N, \psi] \quad \text{for} \quad \psi \in {}^*\mathrm{Conn}(N).$$

   Then ${}^*\mathrm{Conn}(N)$ is a dg B-module and it also has Lie algebra strucre.

Fundamental exact sequence

- Corresponding a D-connection ψ to D, we have a mapping

$$ {}^*\mathrm{Conn}(N) \xrightarrow{\nu} {}^*\mathrm{Der}_A(B) $$

which is a dg B-module homomorphism and at the same time Lie algebra map.

Theorem

There is a short exact sequence of dg B-modules;

$$ 0 \longrightarrow {}^*\mathrm{End}_B(N) \longrightarrow {}^*\mathrm{Conn}(N) \xrightarrow{\nu} {}^*\mathrm{Der}_A(B) \longrightarrow 0 $$

Main theorem

The natural image of
$[\omega_N] \in \mathrm{Ext}^1_B(N, N \otimes_B J) = \mathrm{Hom}_{K(B)}(N, N \otimes_B J(-1))$ into
$[\omega_N] \in \mathrm{Ext}^1_B(N, N \otimes_B J/J^2) = \mathrm{Hom}_{K(B)}(N, N \otimes_B J/J^2(-1))T$ is
called the (classical) Atiyah class of N.

Theorem (Nasseh-Ono-Yoshino,2024)

Let B be a smooth algebra and N be a non-negatively graded semi-free dg B-module.

1. $[\omega_N] = 0$ in $\mathrm{Ext}^1_B(N, N \otimes_B J/J^2)$
2. The fundamental short exact sequence

$$0 \longrightarrow {}^*\mathrm{End}_B(N) \longrightarrow {}^*\mathrm{Conn}(N) \xrightarrow{\nu} {}^*\mathrm{Der}_A(B) \longrightarrow 0$$

   is splitting in the category of dg B-modules.
3. ν splits in the derived category $D(B)$.

Main theorem 2

Corollary

If N is naïvely liftable, then the fundamental short exact sequence for N splits.

Theorem (yet suspended)

Let $A \to B$ be a dg R-algebra homomorphism of commutative dg R-algebras. Assume B is smooth over A, i.e. J/J^2 is a semif-free dg B-module ($+\alpha$ condition). Let N be a semi-free dg B-module ($+\alpha$ condition).
If $\operatorname{Ext}_B^n(N, N) = 0$ for $n > 0$, then the fundamental short exact sequence for N splits.

Thanks for your attention.