Singularity categories and silting objects

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Main Theorem

The singularity category of Λ has no non-zero silting object, if Λ has finite right selfinjective dimension.

Here, Λ denotes a finite dimensional algebra over a field.

① Motivation

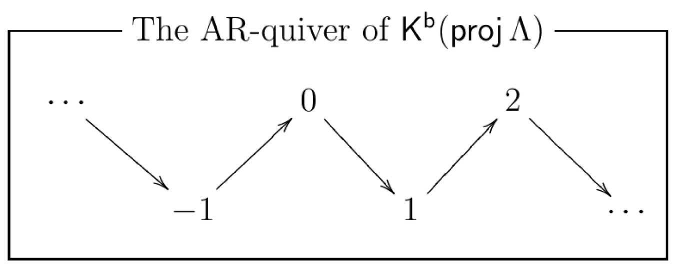
- We know tilting/silting objects are very important in the representation theory of algebras:
  - [in terms of torsion theory] to investigate the structure of derived categories.
  - [in terms of Morita theory] Keller's Theorem and Rickard's Theorem
- So, one wants to get many silting objects.
- However,….. does there always exist a silting object?
- Let us discuss what triangulated category admits **NO** non-zero silting object.

② Definition

Let $\mathcal{T}$ be a triangulated category.
We say that an object T of $\mathcal{T}$ is *silting (tilting)* if
(1) $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any $i > 0$ ($i \neq 0$);
(2) It generates $\mathcal{T}$ by taking direct summands, mapping cones and shifts.

③ Examples of silting objects

We give examples, but which are not our wishes.
$\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ is the usual, and this category has a trivial tilting object Λ.
Let Λ be a path algebra presented by $\bullet \longrightarrow \bullet$.



- $\{i \oplus (i+1) \mid i \in \mathbb{Z}\}$ is the set of tilting objects.
- $\{i \oplus j \mid 0 < j - i \equiv 1 \pmod 3\}$ is that of silts.

What triangulated category should we observe?

④ Prototype

The following are prototypes [Aihara-Iyama]:

1. The bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ of Λ admits a silting object if and only if Λ has finite global dimension.
2. Let Λ be a non-semisimple selfinjective algebra. Then the stable module category $\underline{\mathsf{mod}}\,\Lambda$ of Λ has no non-zero silting object.

Inspired by these facts,
we spotlight the singularity category

$$\mathsf{D}_{\mathsf{sg}}(\Lambda) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$$

Indeed, in the above,
1. $\mathsf{D}_{\mathsf{sg}}(\Lambda) = 0$ and 2. $\mathsf{D}_{\mathsf{sg}}(\Lambda) \simeq \underline{\mathsf{mod}}\,\Lambda$ [Rickard].

⑤ Idea of proof of Main Theorem

To prove the theorem, silting reduction plays a crucial role: [Aihara-Iyama, Iyama-Yang] Let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$ with a silting object T and *satisfying some conditions*. Then there exists a one-to-one correspondence

$$\{\text{silting objects of } \mathcal{T} \text{ with } T \text{ as a direct summand}\}$$
$$\updownarrow \; 1\text{–}1$$
$$\{\text{silting objects of the quotient category } \mathcal{T}/\mathcal{S}\}$$

The assumption of the theorem is needed for $\mathcal{S} = \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ to *satisfy some conditions* in $\mathcal{T} = \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Lambda)$ and $T = \Lambda$.

⑥ Corollary

(1) Let Λ be a non-semisimple Iwanaga-Gorenstein algebra. Then the stable category of maximal Cohen-Macaulay modules has no non-zero silting object.
(Thanks to Buchweitz's theorem, we obtain $\underline{\mathsf{CM}}\,\Lambda \simeq \mathsf{D}_{\mathsf{sg}}(\Lambda)$.)
(2) Assume that Λ has finite right selfinjective dimension. Then $\mathsf{D}_{\mathsf{sg}}(\Lambda^{\mathrm{op}})$ also admits no non-zero tilting object.
Here, Λ^{op} stands for the opposite algebra of Λ.

⑦ Conjecture

We hope:

The singularity category of a finite dimensional algebra always admits no non-zero silting object.

⑧ Remark

Let us consider a triangulated category with a silting object. Then we can obtain infinitely many (non-trivial) silting objects by silting mutation [Aihara-Iyama].
("Hom-finiteness" is needed for a given triangulated category.)