BASS NUMBERS AND BETTI NUMBERS OF COMPLEXES OVER COMMUTATIVE NOETHERIAN RINGS

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Let R be a commutative ring, and let $0 \to R \xrightarrow{\varepsilon} E^0 \to E^1 \to \cdots$ be a minimal injective resolution of a free module R of one generator, and $E^{\cdot}$ the complex $E^0 \to E^1 \to \cdots \to E^n \to \cdots$. Bass studied injective resolutions of left Noetherian rings, in particular, of commutative Noetherian rings [B1], [B2]. In the case of commutative Noetherian rings, he introduced Bass numbers of R-modules as appearance numbers of injective indecomposable R-modules in their minimal injective resolutions, and showed relations between the Bass numbers of rings and Gorenstein rings [B2]. Roberts generalized Bass numbers and Betti numbers to the case of complexes of R-modules, and studied relations between them using dualizing complexes [Ro]. He also applied them to the new intersection conjecture [Ro]. In the case of noncommutative Noetherian rings, we studied $E^{\cdot}$ and injective resolutions of left R-modules and relations between the Auslander-Gorenstein property and the flat dimensions of R-modules [IM], [Mi]. In these papers, from the point of view of derived categories we approached them. In this paper, using techniques of derived categories we study relations between Bass numbers and Betti numbers of complexes in the case of commutative Noetherian local rings.

First, for complexes $X^{\cdot}$ and $Y^{\cdot}$, we study Bass numbers and Betti numbers of $\boldsymbol{R}\operatorname{Hom}^{\cdot}_R(X^{\cdot}, Y^{\cdot})$ and $X^{\cdot} \otimes^{\boldsymbol{L}}_R Y^{\cdot}$, respectively. Then we have a result which contains one of [Ro] 3.6 Theorem (Theorem 2.4). Second, we study Bass numbers of complexes of finite projective dimension and Betti numbers of complexes of finite injective dimension (Theorem 2.8). Then we have more detailed data than Auslander-Buchsbaum Theorem with respect to projective dimension and depth (Corollary 2.9).

1. Preliminaries

In this section, we recall basic techniques of double complexes and methods of derived categories in order to use them in Section 2. For a commutative ring R, let $\mathsf{Mod}\, R$ be the category of R-modules, $\mathsf{Proj}\, R$ (resp., $\mathsf{proj}\, R$) the additive category of projective (resp., finitely generated projective) R-modules, and $\mathsf{Inj}\, R$ the additive category of injective R-modules. We denote by $\mathsf{C}(\mathsf{Mod}\, R)$ (resp., $\mathsf{C}^+(\mathsf{Mod}\, R)$, $\mathsf{C}^-(\mathsf{Mod}\, R)$, $\mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, R)$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathsf{Mod}\, R$. An auto-equivalence $T : \mathsf{C}(\mathsf{Mod}\, R) \to \mathsf{C}(\mathsf{Mod}\, R)$ is called translation if $(TX^{\cdot})^n = X^{n+1}$ and $(Td_X)^n = -d_X^{n+1}$ for any complex $X^{\cdot} = (X^n, d_X^n)$. Sometimes, we write $X^{\cdot}[i]$ for $T^i(X^{\cdot})$. For an additive category $\mathcal{B}$, $\mathsf{K}^*(\mathcal{B})$ is the homotopy category of $\mathcal{B}$. $\mathsf{D}^*(\mathsf{Mod}\, R)$ is the derived category of $\mathsf{Mod}\, R$, and $\mathsf{D}^*_c(\mathsf{Mod}\, R)$ is the full subcategory

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of $\mathsf{D}^*(\mathsf{Mod}\, R)$ consisting of complexes whose cohomologies are finitely presented R-modules, where $* =$ nothing, $+, -, b$.

For $X^\cdot, Y^\cdot \in \mathsf{C}^*(\mathsf{Mod}\, R)$, a morphism $u \in \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, R)}(X^\cdot, Y^\cdot)$ is called a *quasi-isomorphism* if $\mathrm{H}^n(u)$ are isomorphisms for all $n \in \mathbb{Z}$, where $* =$ nothing, $+, -, b$. A complex $X^\cdot = (X^n, d_X^n)$ is called an *acyclic* complex if $\mathrm{H}^n(X^\cdot) = 0$ for all $n \in \mathbb{Z}$.

Definition 1.1. A *double complex* $C^{\cdot\cdot}$ is a bigraded R-module $(C^{p,q})_{p,q\in\mathbb{Z}}$ of $\mathsf{Mod}\, R$ together with $d_{\mathrm{I}}^{p,q} : C^{p,q} \to C^{p+1,q}$ and $d_{\mathrm{II}}^{p,q} : C^{p,q} \to C^{p,q+1}$ such that

$$\begin{aligned} C^{\cdot q} &= (C^{p,q}, d_{\mathrm{I}}^{p,q} : C^{p,q} \to C^{p+1,q}) \\ C^{p\cdot} &= (C^{p,q}, d_{\mathrm{II}}^{p,q} : C^{p,q} \to C^{p,q+1}) \end{aligned}$$

are complexes satisfying $d_{\mathrm{I}}^{p,q+1} d_{\mathrm{II}}^{p,q} + d_{\mathrm{II}}^{p+1,q} d_{\mathrm{I}}^{p,q} = 0$.

A *morphism f of complexes* $X^{\cdot\cdot}$ to $Y^{\cdot\cdot}$ is a collection of morphisms $f^{p,q} : X^{p,q} \to Y^{p,q}$ such that $f^{\cdot q} : X^{\cdot q} \to Y^{\cdot q}$ and $f^{p\cdot} : X^{p\cdot} \to Y^{p\cdot}$ are morphisms of complexes for all $p, q \in \mathbb{Z}$.

We denote by $\mathsf{C}^2(\mathsf{Mod}\, R)$ the category of double complexes of $\mathsf{Mod}\, R$.

Definition 1.2. For a double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}^{p,q}, d_{\mathrm{II}}^{p,q})$, we define the following truncations:

$$(\sigma_{>n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} O \text{ if } q < n \\ \mathrm{Im}\, d_{\mathrm{II}}^{p,q} \text{ if } q = n \\ X^{p,q} \text{ if } q > n \end{cases} \quad (\sigma_{\leq n}^{\mathrm{II}} X^{\cdot\cdot})^{p,q} = \begin{cases} X^{p,q} \text{ if } q < n \\ \mathrm{Ker}\, d_{\mathrm{II}}^{p,q} \text{ if } q = n \\ O \text{ if } q > n \end{cases}$$

Definition 1.3. For a double complex $X^{\cdot\cdot} = (X^{p,q}, d_{\mathrm{I}}^{p,q}, d_{\mathrm{II}}^{p,q})$, we define the *total complexes*

$$\begin{aligned} \mathrm{Tot}\, C^{\cdot\cdot} &= (X^n, d^n), \text{where } X^n = \coprod\nolimits_{p+q=n} C^{p,q}, d^n = \coprod\nolimits_{p+q=n} d_{\mathrm{I}}^{p,q} + d_{\mathrm{II}}^{p,q} \\ \overset{\wedge}{\mathrm{Tot}} C^{\cdot\cdot} &= (Y^n, d^n), \text{where } Y^n = \prod\nolimits_{p+q=n} C^{p,q}, d^n = \prod\nolimits_{p+q=n} d_{\mathrm{I}}^{p,q} + d_{\mathrm{II}}^{p,q}. \end{aligned}$$

Definition 1.4. Let $X^\cdot$, $Y^\cdot$ be complexes of R-modules. We define the double complex $X^\cdot \overset{\cdot\cdot}{\otimes}_R Y^\cdot$ by

$$\begin{aligned} X^\cdot \overset{p,q}{\otimes}_R Y^\cdot &= X^p \otimes_R Y^q, \\ d^{p,q}_{\mathrm{I}\, X^\cdot \overset{\cdot\cdot}{\otimes}_R Y^\cdot} &= d_X^p \otimes_R Y^q, \\ d^{p,q}_{\mathrm{II}\, X^\cdot \overset{\cdot\cdot}{\otimes}_R Y^\cdot} &= (-1)^{p+q} X^p \otimes_R d_Y^q \end{aligned}$$

and define the complex $X^\cdot \overset{\cdot}{\otimes}_R Y^\cdot$ by $\mathrm{Tot}\, X^\cdot \overset{\cdot\cdot}{\otimes}_R Y^\cdot$.

And we define the double complex $\mathrm{Hom}^{\cdot\cdot}_{\mathcal{A}}(X^\cdot, Y^\cdot)$ by

$$\begin{aligned} \mathrm{Hom}_R^{p,q}(X^\cdot, Y^\cdot) &= \mathrm{Hom}_R(X^{-p}, Y^q), \\ d^{p,q}_{\mathrm{I}\ \mathrm{Hom}^{\cdot\cdot}_R(X^\cdot, Y^\cdot)} &= \mathrm{Hom}_R^{p,q}(d_{X^\cdot}^{-p-1}, Y^q), \\ d^{p,q}_{\mathrm{II}\ \mathrm{Hom}^{\cdot\cdot}_R(X^\cdot, Y^\cdot)} &= (-1)^{p+q+1}\, \mathrm{Hom}_R^{p,q}(X^{-p}, d_{Y^\cdot}^q), \end{aligned}$$

and define the complex $\mathrm{Hom}^{\cdot}_R(X^\cdot, Y^\cdot)$ by $\overset{\wedge}{\mathrm{Tot}}\, \mathrm{Hom}^{\cdot\cdot}_R(X^\cdot, Y^\cdot)$.

Definition 1.5. For $u \in \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\, R)}(X^\cdot, Y^\cdot)$, the *mapping cone* of u is a complex $\mathrm{M}^\cdot(u)$ with

$$\begin{aligned} \mathrm{M}^n(u) &= X^{n+1} \oplus Y^n, \\ d^n_{\mathrm{M}^\cdot(u)} &= \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_X^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}. \end{aligned}$$

In this case, we have a distinguished triangle in $\mathsf{K}(\mathsf{Mod}\,R)$

$$X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \mathrm{M}^{\bullet}(u) \xrightarrow{[1\ 0]} X^{\bullet}[1].$$

Definition 1.6. Let R be a commutative ring. A complex $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$ is of *finite projective dimension* (resp., *finite injective dimension*) if there is a complex $Q^{\bullet} \in \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\,R)$ (resp., $Q^{\bullet} \in \mathsf{K}^{\mathrm{b}}(\mathsf{Inj}\,R)$) such that $X^{\bullet} \cong Q^{\bullet}$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,R)$. $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fpd}}$ (resp., $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fid}}$) is the full subcategory of $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$ consisting of complexes of finite projective dimension (resp., finite injective dimension). Then $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fpd}}$ is the category of perfect complexes.

Lemma 1.7. *Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. Then the following hold.*

1. *The following are equivalent for $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$.*
   (a) $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fpd}}$.
   (b) For any $Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$, there is an integer n such that
   $$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,R)}(X^{\bullet}, Y^{\bullet}[i]) = 0$$
   for all $i > n$.
2. *The following are equivalent for $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$.*
   (a) $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fid}}$.
   (b) For any $Y^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$, there is an integer n such that
   $$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,R)}(Y^{\bullet}, X^{\bullet}[i]) = 0$$
   for all $i > n$.

Proof. 1. It is well known that the projective dimension

$$\mathrm{pdim}\, M = \sup\{i \mid \mathrm{Ext}^{i}_{R}(M, k) \neq 0\}$$

for a finitely generated R-module M (see e.g. [BH]). Then it is easy to see that $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fpd}}$ if and only if there is an integer n such that

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,R)}(X^{\bullet}, k[i]) = 0$$

for all $i > n$. Therefore we have the statement.

2. It is well known that the injective dimension

$$\mathrm{idim}\, M = \sup\{i \mid \mathrm{Ext}^{i}_{R}(k, M) \neq 0\}$$

for a finitely generated R-module M (see e.g. [BH]). Then it is easy to see that $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fid}}$ if and only if there is an integer n such that

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,R)}(k, X^{\bullet}[i]) = 0$$

for all $i > n$. Hence we have the statement. □

The notion of *dualizing complexes* was introduced by Grothendieck [RD]. In the case of commutative Noetherian rings, a dualizing complex $U^{\bullet}$ induces the duality $D = \boldsymbol{R}\mathrm{Hom}^{\bullet}_{R}(-, U^{\bullet}) : \mathsf{D}^{*}_{c}(\mathsf{Mod}\,R) \to \mathsf{D}^{\#}_{c}(\mathsf{Mod}\,R)$, where $(*, \#) =$(nothing, nothing), $(+, -), (-, +)$ or (b, b).

Proposition 1.8. *Let R be a commutative Noetherian local ring, $U^{\bullet}$ a dualizing complex for R. Then a duality $D = \boldsymbol{R}\mathrm{Hom}^{\bullet}_{R}(-, U^{\bullet}) : \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R) \to \mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)$ induces the duality between $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fpd}}$ and $\mathsf{D}^{\mathrm{b}}_{c}(\mathsf{Mod}\,R)_{\mathrm{fid}}$.*

Proof. According to Lemma 1.7, $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fid}}$ is the full subcategory of $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$ satisfying the dual categorical property of $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fpd}}$ in $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$. Since $D : \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R) \to \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$ is a duality, it is clear. □

2. Bass Numbers and Betti Numbers of Complexes

According to [Ro], Bass numbers and Betti numbers of complexes are defined as follows.

Definition 2.1. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. For a complex $X^{\bullet} \in \mathsf{D}_c(\mathsf{Mod}\, R)$, the i-th Betti number of $X^{\bullet}$ is defined by

$$\beta_i(X^{\bullet}) = \dim_k \mathrm{H}^{-i}(k \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_R X^{\bullet}),$$

and the i-th Bass number of $X^{\bullet}$ is defined by

$$\mu_i(X^{\bullet}) = \dim_k \mathbf{R}^i \operatorname{Hom}^{\bullet}_R(k, X^{\bullet}).$$

Then for a finitely generated R-module M, the ordinary Bass numbers and the ordinary Betti numbers of M are equal to the numbers of the above definitions.

Definition 2.2. Let R be a commutative Noetherian local ring. A complex $P^{\bullet} = (P^n, d^n_P) \in \mathsf{K}^-(\mathsf{proj}\, R)$ is called a *minimal projective complex* if the induced morphism $P^n \to \operatorname{Cok} d^{n-1}_P$ is an essential epimorphism for any n.

Similarly, a complex $I^{\bullet} = (I^n, d^n_I) \in \mathsf{K}^+(\mathsf{Inj}\, R)$ is called a *minimal injective complex* if the induced morphism $\operatorname{Ker} d^n_P \to I^n$ is an essential monomorphism for any n.

Lemma 2.3. *Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. Then the following hold.*

1. *For $P^{\bullet} = (P^n, d^n_P) \in \mathsf{K}^-(\mathsf{proj}\, R)$, $P^{\bullet}$ is a minimal projective complex if and only if $k \otimes_R d^n_P : k \otimes_R P^n \to k \otimes_R P^{n+1}$ is a zero morphism for any n.*
2. *For $I^{\bullet} = (I^n, d^n_I) \in \mathsf{K}^+(\mathsf{Inj}\, R)$, $I^{\bullet}$ is a minimal injective complex if and only if $\operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), d^n_I \otimes_R R_{\mathfrak{p}}) : \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), I^n_{\mathfrak{p}}) \to \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), I^{n+1}_{\mathfrak{p}})$ is a zero morphism for any $\mathfrak{p} \in \operatorname{Spec} R$ and any n.*

Proof. See [Ro] Chapter I. □

By [RD] Chapter V §3, in the case of commutative Noetherian local rings, for $U^{\bullet} \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$, $U^{\bullet}$ is a dualizing complex if and only if there is an integer n such that $\mu_i(U^{\bullet}) = \delta_{ni}$ (i.e. $\delta_{nn} = 1$ and $\delta_{ni} = 0$ if $i \neq n$). Concerning to relations between Bass numbers and Betti numbers of complexes, we have the following result which contains one of [Ro] 3.6 Theorem.

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. Then the following hold.*

1. *For $X^{\bullet}, Y^{\bullet} \in \mathsf{D}^-_c(\mathsf{Mod}\, R)$, we have*

$$\beta_n(X^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_R Y^{\bullet}) = \sum_{p+q=n} \beta_p(X^{\bullet}) \beta_q(Y^{\bullet})$$

for any integer n.

2. *For* $X^{\bullet} \in \mathsf{D}_c^-(\mathsf{Mod}\, R)$ *and* $Y^{\bullet} \in \mathsf{D}_c^+(\mathsf{Mod}\, R)$*, we have*

$$\mu_n(\boldsymbol{R}\operatorname{Hom}^{\bullet}_R(X^{\bullet}, Y^{\bullet})) = \sum_{p+q=n} \beta_p(X^{\bullet})\mu_q(Y^{\bullet})$$

for any integer n.

In particular, if $Y^{\bullet}$ *is a dualizing complex with* $\mu_i(Y^{\bullet}) = \delta_{0i}$*, then we have*

$$\mu_n(\boldsymbol{R}\operatorname{Hom}^{\bullet}_R(X^{\bullet}, Y^{\bullet})) = \beta_n(X^{\bullet}).$$

Proof. 1. We take minimal projective complexes $P^{\bullet}, Q^{\bullet} \in \mathsf{K}^-(\mathsf{proj}\, R)$ such that $P^{\bullet} \cong X^{\bullet}$ and $Q^{\bullet} \cong Y^{\bullet}$ in $\mathsf{D}(\mathsf{Mod}\, R)$. Then we have isomorphisms in $\mathsf{D}(\mathsf{Mod}\, R)$

$$\begin{aligned} k \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_R(X^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_R Y^{\bullet}) &\cong k\otimes_R(P^{\bullet} \overset{\bullet}{\otimes}_R Q^{\bullet}) \\ &\cong (k\otimes_R P^{\bullet}) \overset{\bullet}{\otimes}_k (k\otimes_R Q^{\bullet}). \end{aligned}$$

Considering the double complex $(k\otimes_R P^{\bullet}) \overset{\bullet\bullet}{\otimes}_k (k\otimes_R Q^{\bullet})$, by Lemma 2.3

$$d^{p,q}_{\mathrm{I}(k\otimes_R P^{\bullet})\overset{\bullet\bullet}{\otimes}_k(k\otimes_R Q^{\bullet})} = (k\otimes_R d_P^p)\otimes_k(k\otimes_R Q^q)$$

is a zero morphism for any p, q, because $P^{\bullet}\otimes_k(k\otimes_R Q^q)$ is a minimal projective complex. Similarly,

$$d^{p,q}_{\mathrm{II}(k\otimes_R P^{\bullet})\overset{\bullet\bullet}{\otimes}_k(k\otimes_R Q^{\bullet})} = (-1)^{p+q}(k\otimes_R P^p)\otimes_k(k\otimes_R d_Q^q)$$

is a zero morphism for any p, q. Therefore by taking cohomologies, we have

$$\begin{aligned} \mathrm{H}^n(k\otimes_R(P^{\bullet} \overset{\bullet}{\otimes}_R Q^{\bullet})) &\cong \bigoplus_{p+q=n} (k\otimes_R P^p)\otimes_k(k\otimes_R Q^q) \\ &\cong \bigoplus_{p+q=n} \mathrm{H}^p(k\otimes_R P^{\bullet})\otimes_k \mathrm{H}^q(k\otimes_R Q^{\bullet}), \end{aligned}$$

because $\mathrm{H}^p(k\otimes_R P^{\bullet}) \cong k\otimes_R P^p$ and $\mathrm{H}^q(k\otimes_R Q^{\bullet}) \cong k\otimes_R Q^q$. Since

$$\dim_k \mathrm{H}^p(k\otimes_R P^{\bullet})\otimes_k \mathrm{H}^q(k\otimes_R Q^{\bullet}) = \dim_k \mathrm{H}^p(k\otimes_R P^{\bullet}) \dim_k \mathrm{H}^q(k\otimes_R Q^{\bullet}),$$

we have the statement.

2. We take a minimal projective complex $P^{\bullet} \in \mathsf{K}^-(\mathsf{proj}\, R)$ and a minimal injective complex $Q^{\bullet} \in \mathsf{K}^+(\mathsf{Inj}\, R)$ such that $P^{\bullet} \cong X^{\bullet}$ and $Q^{\bullet} \cong Y^{\bullet}$ in $\mathsf{D}(\mathsf{Mod}\, R)$. Since $\operatorname{Hom}^{\bullet}_R(P^{\bullet}, Q^{\bullet}) \in \mathsf{K}^+(\mathsf{Inj}\, R)$, we have isomorphisms in $\mathsf{D}(\mathsf{Mod}\, R)$

$$\begin{aligned} \boldsymbol{R}\operatorname{Hom}^{\bullet}_R(k, \boldsymbol{R}\operatorname{Hom}^{\bullet}_R(X^{\bullet}, Y^{\bullet})) &\cong \operatorname{Hom}_R(k, \operatorname{Hom}^{\bullet}_R(P^{\bullet}, Q^{\bullet})) \\ &\cong \operatorname{Hom}^{\bullet}_R(k\otimes_R P^{\bullet}, Q^{\bullet}). \end{aligned}$$

Consider the double complex $\operatorname{Hom}^{\bullet\bullet}_R(k\otimes_R P^{\bullet}, Q^{\bullet})$. According to Lemma 2.3,

$$d^{p,q}_{\mathrm{II}\operatorname{Hom}^{\bullet\bullet}_R(k\otimes_R P^{\bullet}, Q^{\bullet})} = (-1)^{p+q+1}\operatorname{Hom}_R(k\otimes_R P^{-p}, d_Q^q)$$

is a zero morphism for any p, q, because $Q^{\bullet}$ is a minimal injective complex. Similarly,

$$d^{p,q}_{\mathrm{I}\operatorname{Hom}^{\bullet\bullet}_R(k\otimes_R P^{\bullet}, Q^{\bullet})} = \operatorname{Hom}_R(k\otimes_R d_P^{-p-1}, Q^q)$$

is a zero morphism for any p, q. Therefore by taking cohomologies, we have

$$\begin{aligned} \mathrm{H}^n(\operatorname{Hom}^{\bullet}_R(k\otimes_R P^{\bullet}, Q^{\bullet})) &\cong \bigoplus_{p+q=n} \operatorname{Hom}_R(k\otimes_R P^{-p}, Q^q) \\ &\cong \bigoplus_{p+q=n} \operatorname{Hom}_R(\mathrm{H}^{-p}(k\otimes_R P^{\bullet}), Q^q), \end{aligned}$$

because $\mathrm{H}^{-p}(k\otimes_R P^{\boldsymbol{\cdot}}) \cong k\otimes_R P^{-p}$. Since

$$\dim_k \mathrm{Hom}_R(\mathrm{H}^{-p}(k\otimes_R P^{\boldsymbol{\cdot}}), Q^q) = \beta_p(P^{\boldsymbol{\cdot}})\mu_q(Q^{\boldsymbol{\cdot}}),$$

we complete the proof. □

Lemma 2.5. *For a double complex $X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} = (X^{p,q}, d_{\mathrm{I}}, d_{\mathrm{II}})$, the following hold.*

1. *If $X^{p,q} = O$ for $q < m, n < q$ for $m \leq n$ and $X^{p\boldsymbol{\cdot}}$ are acyclic complexes in $\mathsf{C}(\mathsf{Mod}\, R)$ for all p, then* Tot $X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ *is acyclic in* $\mathsf{C}(\mathsf{Mod}\, R)$.
2. *If $X^{p,q} = O$ for $q < n$ and $X^{p\boldsymbol{\cdot}}$ are acyclic complexes in $\mathsf{C}(\mathsf{Mod}\, R)$ for all p, then* Tot $X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ *is acyclic in* $\mathsf{C}(\mathsf{Mod}\, R)$.

Proof. 1. Let $n_X = n - m$. We have an exact sequence in $\mathsf{C}^2(\mathsf{Mod}\, R)$

$$O \to \sigma^{\mathrm{II}}_{\leq n-1} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to \sigma^{\mathrm{II}}_{>n-1} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to O.$$

Then we have the exact sequence in $\mathsf{C}(\mathsf{Mod}\, R)$

$$O \to \mathrm{Tot}\ \sigma^{\mathrm{II}}_{\leq n-1} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to \mathrm{Tot}\ X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to \mathrm{Tot}\ \sigma^{\mathrm{II}}_{>n-1} X^{\boldsymbol{\cdot} n} \to O.$$

By the assumption of induction on n_X, Tot $\sigma^{\mathrm{II}}_{\leq n-1} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ is acyclic. It is easy to see that Tot $\sigma^{\mathrm{II}}_{>n-1} X^{\boldsymbol{\cdot} n} \cong \mathrm{M}^{\boldsymbol{\cdot}}(1_{X^{\boldsymbol{\cdot} n}})[-n]$ is acyclic. Then Tot $X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ is acyclic.

2. Since we have the canonical monomorphisms in $\mathsf{C}^2(\mathsf{Mod}\, R)$

$$\sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \xrightarrow{g_r} \sigma^{\mathrm{II}}_{\leq r+1} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}},$$

we have an exact sequence in $\mathsf{C}(\mathsf{Mod}\, R)$

$$O \to \coprod_r \mathrm{Tot}\ \sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \xrightarrow{1-\mathrm{shift}} \coprod_r \mathrm{Tot}\ \sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to \varinjlim \mathrm{Tot}\ \sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \to O,$$

and

$$\varinjlim \mathrm{Tot}\, \sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}} \cong \mathrm{Tot}\, X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}.$$

By 1, Tot $\sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ is acyclic for all r. By taking cohomologies of the above exact sequence, $\varinjlim \mathrm{Tot}\, \sigma^{\mathrm{II}}_{\leq r} X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$ is acyclic, and hence so is Tot $X^{\boldsymbol{\cdot}\boldsymbol{\cdot}}$. □

Proposition 2.6. *Let R be a commutative ring, $R \xrightarrow{\varepsilon} E^{\boldsymbol{\cdot}}$ an injective resolution, and $P^{\boldsymbol{\cdot}} \in \mathsf{K}(\mathsf{Proj}\, R)$. Then there is a quasi-isomorphism $P^{\boldsymbol{\cdot}} \to P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}}$.*

Proof. Since $P^{\boldsymbol{\cdot}}$ is isomorphic to $P^{\boldsymbol{\cdot}}\otimes_R R$, a morphism $\varepsilon : R \to E^{\boldsymbol{\cdot}}$ induces the morphism $\varepsilon' : P^{\boldsymbol{\cdot}} \to P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}}$ of complexes. Let $F^{\boldsymbol{\cdot}}$ be the complex $R \xrightarrow{\varepsilon} E^0 \to E^1 \to \cdots$, where the term R is the (-1)-th term. It is easy to see that the mapping cone $\mathrm{M}^{\boldsymbol{\cdot}}(\varepsilon')$ is isomorphic to $P^{\boldsymbol{\cdot}} \dot{\otimes}_R F^{\boldsymbol{\cdot}}$. Then we have a distinguished triangle in $\mathsf{K}(\mathsf{Mod}\, R)$

$$P^{\boldsymbol{\cdot}} \to P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}} \to P^{\boldsymbol{\cdot}} \dot{\otimes}_R F^{\boldsymbol{\cdot}} \to P^{\boldsymbol{\cdot}}[1].$$

Since $P^p \dot{\otimes}_R F^{\boldsymbol{\cdot}}$ is acyclic for every p, by Lemma 2.5 $P^{\boldsymbol{\cdot}} \dot{\otimes}_R F^{\boldsymbol{\cdot}}$ is acyclic, and hence we complete the proof. □

Lemma 2.7. *Let R be a commutative Noetherian local ring, $X^{\boldsymbol{\cdot}} \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fpd}}$. Then we have*

$$\beta_n(\boldsymbol{R}\mathrm{Hom}_R(X^{\boldsymbol{\cdot}}, R)) = \beta_{-n}(X^{\boldsymbol{\cdot}})$$

for any integer n.

Proof. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, R)$ such that $P^\bullet \cong X^\bullet$ in $\mathsf{D}(\mathsf{Mod}\, R)$. Then we have isomorphisms

$$\begin{aligned} k \otimes^{\boldsymbol{L}}_R \boldsymbol{R}\operatorname{Hom}_R(X^\bullet, R) &\cong k \otimes_R \operatorname{Hom}_R(P^\bullet, R) \\ &\cong \operatorname{Hom}_R(P^\bullet, k) \\ &\cong \operatorname{Hom}_k(k \otimes_R P^\bullet, k) \\ &\cong \operatorname{Hom}_k(k \otimes^{\boldsymbol{L}}_R X^\bullet, k). \end{aligned}$$

Hence we have the statement. □

Theorem 2.8. *Let R be a commutative Noetherian local ring. Then the following hold.*

1. *For $X^\bullet$ a complex in $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fpd}}$, we have*
$$\mu_n(X^\bullet) = \sum_{q-p=n} \beta_p(X^\bullet)\mu_q(R)$$
for any integer n.
2. *For $X^\bullet$ a complex in $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fid}}$, we have*
$$\beta_n(X^\bullet) = \sum_{q-p=n} \mu_p(X^\bullet)\mu_q(R)$$
for any integer n.

Proof. 1. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, R)$ such that $P^\bullet \cong X^\bullet$ in $\mathsf{D}(\mathsf{Mod}\, R)$. According to Proposition 2.6, $P^\bullet$ is quasi-isomorphic to $P^\bullet \dot{\otimes}_R E^\bullet$. Since $P^\bullet \in \mathsf{K}^{+}(\mathsf{proj}\, R)$, $P^\bullet \dot{\otimes}_R E^\bullet$ is in $\mathsf{K}^{+}(\mathsf{Inj}\, R)$. Then we have isomorphisms in $\mathsf{D}(\mathsf{Mod}\, R)$

$$\begin{aligned} \boldsymbol{R}\operatorname{Hom}^\bullet_R(k, X^\bullet) &\cong \operatorname{Hom}_R(k, P^\bullet \dot{\otimes}_R E^\bullet) \\ &\cong \operatorname{Hom}_R(k, \operatorname{Hom}^\bullet_R(\operatorname{Hom}_R(P^\bullet, R), E^\bullet)) \\ &\cong \boldsymbol{R}\operatorname{Hom}_R(k, \boldsymbol{R}\operatorname{Hom}^\bullet_R(\boldsymbol{R}\operatorname{Hom}_R(X^\bullet, R), E^\bullet)). \end{aligned}$$

Since $\boldsymbol{R}\operatorname{Hom}_R(X^\bullet, R) \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$, by Theorem 2.4, we have

$$\mu_n(X^\bullet) = \sum_{p+q=n} \beta_p(\boldsymbol{R}\operatorname{Hom}_R(X^\bullet, R))\mu_q(R).$$

By Lemma 2.7, $\beta_p(\boldsymbol{R}\operatorname{Hom}_R(X^\bullet, R)) = \beta_{-p}(X^\bullet)$, and hence we have the statement.

2. According to [RD] and [Ro], Bass numbers, Betti numbers, finiteness of projective dimension and finiteness of injective dimension of complexes are invariant under the completion of complexes. Then we may assume that R is a complete local ring, and then that R has a dualizing complex $U^\bullet$ with $\mu_i(U^\bullet) = \delta_{0i}$. Let $D = \boldsymbol{R}\operatorname{Hom}^\bullet_R(-, U^\bullet) : \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R) \to \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)$ be a duality. For a complex $X^\bullet$ in $\mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fid}}$, by Proposition 1.8, we have $DX^\bullet \in \mathsf{D}^{\mathrm{b}}_c(\mathsf{Mod}\, R)_{\mathrm{fpd}}$. By 1, we have

$$\mu_n(DX^\bullet) = \sum_{q-p=n} \beta_p(DX^\bullet)\mu_q(R).$$

According to Theorem 2.4, $\mu_n(DX^\bullet) = \beta_n(X^\bullet)$ and $\beta_p(DX^\bullet) = \mu_p(X^\bullet)$, and hence we complete the proof. □

Applying the above theorem to finitely generated R-modules, we have more detailed data than Auslander-Buchsbaum Theorem. For an R-module M, we denote by $\operatorname{pdim} M$ (resp., $\operatorname{idm} M$) the projective dimension (resp., the injective dimension) of M.

Corollary 2.9. *Let R be a commutative Noetherian local ring, and M a finitely generated R-module. Then the following hold.*

1. *If the projective dimension of M is finite, then for any integer n we have*
$$\mu_n(M) = \sum_{q-p=n} \beta_p(M)\mu_q(R).$$
In particular, we have $\operatorname{depth} M + \operatorname{pdim} M = \operatorname{depth} R$.
2. *If the injective dimension of M is finite, then for any integer n we have*
$$\beta_n(M) = \sum_{q-p=n} \mu_p(M)\mu_q(R).$$
In particular, we have $\operatorname{idim} M = \operatorname{depth} R$.

Proof. 1. Let $P^{\boldsymbol{\cdot}} : O \to P^{-s} \to \cdots \to P^{-1} \to P^0 \to O$ be a projective resolution of M, then $\mu_n(M) = \mu_n(P^{\boldsymbol{\cdot}})$. The first assertion is trivial by Theorem 2.8. The above equation implies

$$\begin{aligned}\min\{n|\mu_n(M) \neq 0\} &= \min\{q|\mu_q(R) \neq 0\} - \max\{p|\beta_p(M) \neq 0\}\\ \operatorname{depth} M &= \operatorname{depth} R - \operatorname{pdim} M.\end{aligned}$$

2. Let $I^{\boldsymbol{\cdot}} : O \to I^0 \to I^1 \to \cdots \to I^t \to O$ be an injective resolution of M, then $\beta_n(M) = \beta_n(I^{\boldsymbol{\cdot}})$. The first assertion is trivial by Theorem 2.8. The above equation implies

$$\begin{aligned}0 &= \min\{n|\beta_n(M) \neq 0\}\\ &= \min\{q|\mu_q(R) \neq 0\} - \max\{p|\mu_p(M) \neq 0\}\\ &= \operatorname{depth} R - \operatorname{idim} M.\end{aligned}$$

□

Remark 2.10. In Theorem 2.8, 1, we need that $X^{\boldsymbol{\cdot}}$ is of finite projective dimension. For a complex $P^{\boldsymbol{\cdot}} \in \mathsf{K}^-(\mathsf{proj}\, R)$, by Proposition 2.6, $P^{\boldsymbol{\cdot}}$ is quasi-isomorphic to $P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}}$. In the case of commutative Noetherian rings, $P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}}$ is a complex of injective R-modules. But in general it doesn't satisfy the condition that $\operatorname{Hom}^{\boldsymbol{\cdot}}_R(N^{\boldsymbol{\cdot}}, P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}})$ is acyclic for any acyclic complex $N^{\boldsymbol{\cdot}}$ (see [BN] or [Sp]). Therefore in general we have

$$\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_R(k, P^{\boldsymbol{\cdot}}) \ncong \operatorname{Hom}_R(k, P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}}).$$

Indeed, let R be a commutative Gorenstein local ring of Krull dimension d, M a finitely generated R-module of infinite projective dimension, and $P^{\boldsymbol{\cdot}}$ a minimal projective resolution of M. Then $\mu_n(M) = 0$ for any $n < 0$. But by the proof of Theorem 2.8, we have

$$\dim_k \mathrm{H}^n(\operatorname{Hom}_R(k, P^{\boldsymbol{\cdot}} \dot{\otimes}_R E^{\boldsymbol{\cdot}})) = \beta_{d-n}(M)$$

for any integer n.

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