COHEN-MACAULAY APPROXIMATIONS AND NOETHERIAN ALGEBRAS

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Introduction

The notion of Cohen-Macaulay approximations was introduced by Auslander and Buchweitz, and was studied widely [1]. In the case of commutative complete Cohen-Macaulay local rings, there exist minimal CM-approximations of all finitely generated modules. Then invariant theory with respect to this approximations was studied by several authors. Also, Hashimoto and Shida showed the existence of minimal ones without compltleteness of rings [4]. In non-commutative ring theory, we studied the relation between duality for derived categories and CM-approximations, and studied cotilting bimodules of finite injective dimension as the non-commutative ring version of dualizing modules [6]. Also, we gave the condition of categories for the existence of minimal ones. In this note, we study approximations in case of Noetherian algebras.

First, we study the case that modules of infinite injective dimension induce the theory of CM-approximations. In the case of commutative ring , by [3] and [5], we know that pointwise dualizing modules induce the duality on derived categories of modules. We show that if A is a CM R-algebra over a commutative Cohen-Macaulay ring R with a pointwise dualizing module ω, then the duality between $D^b(\mathrm{mod}A)$ and $D^b(A\text{-mod})$ exists, and then mod$A$ has CM(A_A)-approximations (Theorem 1.2).

Second, we study the existence of minimal ones. In [6], we showed that semiperfectness of some subcategories of modules induces the existnce of minimal ones. We applied it to the case of a finite R-algebra A over a commutative Noetherian local ring R (Corollary 2.2). Furthermore, we showed that if an endomorphism ring of cotilting bimodule is a semiperfect ring, then mod$A$ has minimal rac(U_A)-approximations (Theorem 2.6). Then if A is a CM R-algebra over a commutative Cohen-Macaulay ring R with a dualizing module ω, then mod$A$ has minimal CM(A_A)-approximations (Corollary 2.7).

1. Dualities for Derived Categories and CM-approximations

Throughout this note, we assume that all rings have non-zero unity, and that all modules are unital. For a ring A, we denote by mod$A$ (resp., A-mod) the category of finitely presented right (resp., left) A-modules, and denote by $\mathcal{P}_A$ (resp., ${}_A\mathcal{P}$) the category of finitely generated projective right (resp., left) A-modules. For a right A-module U_A, we denote by addU_A the category of right A-modules which are direct summands of finite direct sums of copies of U_A, and we define the following subcategories of mod$A$:

fres(U_A) := $\{M \in \mathrm{mod}A \mid$ there exist some integer n and $X_i \in \mathrm{add}U_A$ $(0 \leq i \leq n)$ such that $0 \to X_n \to \ldots \to X_0 \to M \to 0$ is exact in mod$A\}$;

cores(U_A) := $\{M \in \mathrm{mod}A \mid$ there exist $X^i \in \mathrm{add}U_A$ $(i \geq 0)$ such that $0 \to M \to X^0 \to \ldots \to X^n \to \ldots$ is exact in mod$A\}$;

rac(U_A) := $\{M \in \mathrm{mod}A \mid \mathrm{Ext}^i_A(M, U_A) = 0$ for all $i > 0\}$.

Let A and B be rings, ${}_BU_A$ a B-A-bimodule. We will call ${}_BU_A$ a cotilting B-A-bimodule provided that it satisfies the following:

(C1) ${}_BU_A$ is finitely presented as both a right A-module and a left B-module;

(C2r) the injective dimension idim U_A of U_A is finite;

(C2l) the injective dimension $\mathrm{idim}_B U$ of ${}_BU$ is finite;

(C3r) $\mathrm{Ext}^i_A(U,U) = 0$ for all $i > 0$; (C3l) $\mathrm{Ext}^i_B(U,U) = 0$ for all $i > 0$;

(C4r) the natural ring morphism $B \to \mathrm{Hom}_A(U,U)$ is an isomorphism;
(C4l) the natural ring morphism $A^{\mathrm{op}} \to \mathrm{Hom}_B(U,U)$ is an isomorphism.
In case of $B = A$, we call a cotilting A-A-bimodule a dualizing A-bimodule.

Let $\mathcal{A}$ be an additive category, $K(\mathcal{A})$ the homotopy category of $\mathcal{A}$, and $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ full subcategories of $K(\mathcal{A})$ generated by the bounded below complexes, the bounded above complexes, the bounded complexes, respectively. For a full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$, let $K^{*,b}(\mathcal{B})$ be a full subcategory of $K^*(\mathcal{B})$ generated by complexes which have bounded homologies, and $K^*(\mathcal{B})_{\mathrm{Qis}}$ a quotient category of $K^*(\mathcal{B})$ by the multiplicative set of quasi-isomorphisms, where $* = +$ or -. We denote $K^*(\mathcal{A})_{\mathrm{Qis}}$ by $D^*(\mathcal{A})$.

We give abbreviated definitions of CM-approximations here (see [1] for more precise definitions). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be full subcategories of mod$A$ such that $\mathrm{Ext}^1_A(\mathfrak{X},\mathfrak{Y}) = 0$. For $M \in$ mod$A$, a short exact sequence: $0 \to Y \to X \xrightarrow{h} M \to 0$ (resp., $0 \to M \xrightarrow{h} Y \to X \to 0$) is called a $\mathfrak{X}$-approximation (resp., $\mathfrak{Y}$-hull) if X and Y belong to $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. The above $\mathfrak{X}$-approximation (resp., $\mathfrak{Y}$-hull) is called minimal if $h \circ f$ (resp., $f \circ h$) is not equal to h for every non-isomorphism f: $X \to X$ (resp., $f: Y \to Y$).

Proposition 1.1. *Let A be a right coherent ring, B a left coherent ring, and ${}_BU_A$ a B-A-bimodule which satisfies the condition* (C1) *such that* $\boldsymbol{R}^b\mathrm{Hom}_A(-, {}_BU_A)$ *and* $\boldsymbol{R}^b\mathrm{Hom}_B(-, {}_BU_A)$ *induce the duality between* $D^b(\mathrm{mod}A)$ and $D^b(B\text{-mod})$. *If* $\mathrm{rac}(U_A) = \mathrm{cores}(U_A)$, *then* $K^{+,b}(\mathrm{add}\,U_A)$ *is equivalent to* $D^b(\mathrm{mod}A)$.

Proof. Since $K^{-,b}({}_B\mathcal{P})$ is equivalent to $D^b(B\text{-mod})$, by the assumption, $\boldsymbol{R}^b\mathrm{Hom}_B(-, {}_BU_A)|_{K^{-,b}({}_B\mathcal{P})} : K^{-,b}({}_B\mathcal{P}) \to D^b(\mathrm{mod}A)$ is a duality and is isomorphic to $\mathrm{Q}\circ\mathrm{Hom}_B(-, {}_BU_A)$, where Q is the natural quotient $K^{+,b}(\mathrm{add}\,U_A) \to D^b(\mathrm{mod}A)$. Then it suffices to show that $\mathrm{Hom}_B(-, {}_BU_A) : K^{-,b}({}_B\mathcal{P}) \to K^{+,b}(\mathrm{add}\,U_A)$ is a duality. Clearly, $\mathrm{Hom}_B(-, {}_BU_A)$ is fully faithful. Let $X^\bullet$ be any complex in $K^{+,b}(\mathrm{add}\,U_A)$. We may assume $X^\bullet$ is the following complex:

$$0 \to U^0 \xrightarrow{d_0} U^1 \xrightarrow{d_1} \dots \to U^n \xrightarrow{d_n} U^{n+1} \to \dots,$$

where all U^i belong to $\mathrm{add}\,U_A$, such that $\mathrm{H}^i(X^\bullet) = 0$ for all $i > n$. Then $\mathrm{Ker}\,d_i$

belongs to cores(U_A) for all $i \geq n$. Since rac(U_A) = cores(U_A), we have the following short exact sequences:

$$0 \to \mathrm{Hom}_B(\mathrm{Ker}d_{n+1}, {}_BU_A) \to \mathrm{Hom}_B(U^n, {}_BU_A) \to \mathrm{Hom}_B(\mathrm{Ker}d_n, {}_BU_A) \to 0,$$
$$0 \to \mathrm{Hom}_B(\mathrm{Ker}d_{n+2}, {}_BU_A) \to \mathrm{Hom}_B(U^{n+1}, {}_BU_A) \to \mathrm{Hom}_B(\mathrm{Ker}d_{n+1}, {}_BU_A) \to 0,$$
$$\ldots.$$

Then $\mathrm{H}^i\mathrm{Hom}_A(X^\bullet, {}_BU_A) = 0$ for all $i < -n$, and therefore $\mathrm{Hom}_A(X^\bullet, {}_BU_A)$ belongs to $K^{-,b}({}_B\mathcal{P})$. Since all U^i belong to addU_A, $\mathrm{Hom}_B(\mathrm{Hom}_A(X^\bullet, {}_BU_A), {}_BU_A)$ is isomorphic to $X^\bullet$ in $D^b(\mathrm{mod}A)$. Hence $\mathrm{Hom}_B(-, {}_BU_A)$ is dense.

Let R be a commutative Cohen-Macaulay ring. A finitely generated R-module ω is called a pointwise dualizing module if ω_p is a dualizing R_p-module for every $\mathfrak{p} \in \mathrm{Spec}\,R$. A finitely generated R-module M is called a maximal Cohen-Macaulay R-module if $\mathrm{depth}_{pR_p} M_p$ is equal to Krull dimension of R_p for all $\mathfrak{p} \in \mathrm{Spec}\,R$. In case that R has a pointwise dualizing module ω, M is a maximal Cohen-Macaulay R-module if and only if $\mathrm{Ext}_R^i(M,\omega) = 0$ for all $i > 0$ (see [1]). An R-algebra A is called a finite R-algebra if A is a finitely generated R-module. We call a finite R-algebra A a CM R-algebra if A is a finitely generated maximal Cohen-Macaulay R-module. For a CM R-algebra A, we denote by CM(A_A) the category of finitely generated right A-modules which are maximal Cohen-Macaulay R-modules.

By [3], if R has a pointwise dualizing module ω, then $\boldsymbol{R}^b\mathrm{Hom}_{R_p}(-, \omega_p)$ induces the duality on $D^b(\mathrm{mod}R_p)$ for every $\mathfrak{p} \in \mathrm{Spec}R$. According to [5], the image of $\boldsymbol{R}^b\mathrm{Hom}_R(-, \omega) : D^b(\mathrm{mod}R) \to D(\mathrm{mod}R)$ is contained in $D^b(\mathrm{mod}R)$. Hence, if R has a pointwise dualizing module ω, then $\boldsymbol{R}^b\mathrm{Hom}_R(-, \omega)$ induces the duality on $D^b(\mathrm{mod}R)$. Furthermore, we have the following example of infinite injective dimension which satisfies the condition of Proposition 1.1.

THEOREM 1.2. *Let R be a commutative Cohen-Macaulay ring with a pointwise dualizing module* ω *, A a CM R-algebra, and* $U = \mathrm{Hom}_R(A,\omega)$. *Then the following hold.*

(a) $\boldsymbol{R}^b\mathrm{Hom}_A(-, U_A)$ *and* $\boldsymbol{R}^b\mathrm{Hom}_A(-, {}_AU)$ *induce the duality between* $D^b(\mathrm{mod}A)$ *and* $D^b(A\text{-mod})$.

(b) $\mathrm{rac}(U_A) = \mathrm{cores}(U_A) = \mathrm{CM}(A_A)$.
(c) *Every finitely generated right A-module has a* $\mathrm{CM}(A_A)$*-approximation and a* $\mathrm{fres}(U_A)$*-hull.*

Proof. (a) Let $I^\bullet$ be an injective resolution of ω in mod$R$. Since A is a finitely generated maximal Cohen-Macaulay R-module, $\mathrm{Hom}_R(A,I^\bullet)$ is an injective resolution of $U = \mathrm{Hom}_R(A, \omega)$. For every complex $X^\bullet \in D^b(\mathrm{mod}A)$, we have the following isomorphisms in $D^b(\mathrm{mod}R)$:

$$\begin{aligned}\boldsymbol{R}^b\mathrm{Hom}_A(X^\bullet, U_A) &\simeq \mathrm{Hom}^\bullet_A(X^\bullet, \mathrm{Hom}_R(A,I^\bullet))\\ &\simeq \mathrm{Hom}^\bullet_R(X^\bullet, I^\bullet)\\ &\simeq \boldsymbol{R}^b\mathrm{Hom}_R(X^\bullet, \omega).\end{aligned}$$

Since $X^\bullet$ belongs to $D^b(\mathrm{mod}R)$, by the proof of [5, Theorem 1], $\boldsymbol{R}^b\mathrm{Hom}_R(X^\bullet, \omega)$ belongs to $D^b(\mathrm{mod}R)$, and then $\boldsymbol{R}^b\mathrm{Hom}_A(X^\bullet, U_A)$ belongs to $D^b(A\text{-mod})$. Therefore the image of $\boldsymbol{R}^b\mathrm{Hom}_A(-, U_A)$ is contained in $D^b(A\text{-mod})$. Similarly, the image of $\boldsymbol{R}^b\mathrm{Hom}_A(-, {}_AU)$ is contained in $D^b(\mathrm{mod}A)$. For every integer i and every $\mathfrak{p} \in \mathrm{Spec}R$, we have the following commutative diagram:

$$\begin{array}{ccc}(\mathrm{H}^iX^\bullet)_p & \to & (\mathrm{H}^i\boldsymbol{R}^b\mathrm{Hom}_A(\boldsymbol{R}^b\mathrm{Hom}_A(X^\bullet, U), U))_p\\ \downarrow & & \downarrow\\ \mathrm{H}^i(X^\bullet_p) & \to & \mathrm{H}^i\boldsymbol{R}^b\mathrm{Hom}_{A_p}(\boldsymbol{R}^b\mathrm{Hom}_{A_p}(X^\bullet_p, U_p), U_p),\end{array}$$

where vertical arrows are isomorphisms. Since ω_p is a dualizing R_p-module for all $\mathfrak{p} \in \mathrm{Spec}R$, by [6, Proposition 2.12], U_p is a dualizing A_p-bimodule. Then the bottom arrow in the above diagram is an isomorphism, and so is the top arrow in the above. Hence the natural morphism $\mathbf{id}_{D^b(\mathrm{mod}A)} \to \boldsymbol{R}^b\mathrm{Hom}_A(\boldsymbol{R}^b\mathrm{Hom}_A(-, U_A), {}_AU)$ is an isomorphism. Similarly, the natural morphism $\mathbf{id}_{D^b(A\text{-mod})} \to \boldsymbol{R}^b\mathrm{Hom}_A(\boldsymbol{R}^b\mathrm{Hom}_A(-, {}_AU), U_A)$ is an isomorphism.
(b) For every $M \in \mathrm{mod}A$, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Ext}^i_A(M,U) &\simeq \mathrm{H}^i\mathrm{Hom}_A(M, \mathrm{Hom}_R(A,I^\bullet))\\ &\simeq \mathrm{H}^i\mathrm{Hom}_R(M,I^\bullet)\end{aligned}$$

$$\simeq \mathrm{Ext}^i_R(M,\omega).$$

Then we have $\mathrm{rac}(U_A) = \mathrm{CM}(A_A)$. According to [6, Theorem 2.10, Corollary 2.11], it is easy to see thet $\mathrm{rac}(U_A)$ is contained in $\mathrm{cores}(U_A)$. Conversely, given $M \in \mathrm{cores}(U_A)$, M_p belongs to $\mathrm{cores}((U_p)_{A_p})$ for all $\mathfrak{p} \in \mathrm{Spec}R$. By [6, Theorem 2.10, Corollary 2.11], M_p belongs to $\mathrm{rac}((U_p)_{A_p})$. For all $\mathfrak{p} \in \mathrm{Spec}R$, we have

$$\mathrm{Ext}^i_A(M, U)_p \simeq \mathrm{Ext}^i_{A_p}(M_p, U_p)$$
$$= 0 \text{ for all } i > 0.$$

Therefore $\mathrm{Ext}^i_A(M, U) = 0$ for all $i > 0$, and hence M belongs to $\mathrm{rac}(U_A)$.

(c) Let M be a finitely generated right A-module. Considering M as a complex in $D^b(\mathrm{mod}A)$, by Proposition 1.1, M is isomorphic to the following complex in $D^b(\mathrm{mod}A)$:

$$0 \to U^{-s} \to \dots \to U^{-1} \xrightarrow{d_{-1}} U^0 \xrightarrow{d_0} U^1 \to U^2 \to \dots \to U^i \to \dots,$$

where $U^i \in \mathrm{add}_B U$ $(-s \leq i)$. Then we have the following short exact sequences:

$$0 \to \mathrm{Im}d_{-1} \to \mathrm{Ker}d_0 \to M \to 0,$$
$$0 \to M \to \mathrm{Cok}d_{-1} \to \mathrm{Im}d_0 \to 0.$$

Furthermore, $\mathrm{Ker}d_0$ and $\mathrm{Im}d_0$ belong to $\mathrm{cores}(U_A)$, and $\mathrm{Cok}d_{-1}$ and $\mathrm{Im}d_{-1}$ belong to $\mathrm{fres}(U_A)$.

By (b), we complete the proof.

Remark. More generally, for a finite R-algebra A, let a complex $U^\bullet$ be $\mathrm{Hom}_R(A, I^\bullet)$, where $I^\bullet$ is an injective resolution of ω. Then, according to the proof of Theorem 1.2 and [7, Corollary 2.14], $\boldsymbol{R}^b\mathrm{Hom}^\bullet_A(-, U^\bullet_A)$ and $\boldsymbol{R}^b\mathrm{Hom}_A(-, {}_AU^\bullet)$ also induce the duality between $D^b(\mathrm{mod}A)$ and $D^b(A\text{-mod})$. Hence we get an example of a dualizing bimodule complex of infinite injective dimension.

2. Minimal CM-Approximations

For an additive category $\mathscr{A}$, we will call $\mathscr{A}$ semiperfect if $\mathrm{End}_{\mathscr{A}}(X)$ is a semiperfect ring for every object $X \in \mathscr{A}$. Let R be a commutative Noetherian local ring, $\hat{R}$ a completion of R with respect to the maximal ideal of R, and A a finite R-algebra. For a finitely generated right A-module M, we denote $M \otimes_R \hat{R}$ by $\hat{M}$.

THEOREM 2.1 [6]. *Let A be a right coherent ring , B a left coherent ring, and ${}_BU_A$ a B-A-bimodule satisfying the conditions of Proposition 1.1.*

(a) *If* $\mathrm{rac}(U_A)$ *is semiperfect, then there exists a unique minimal* $\mathrm{rac}(U_A)$-*approximation in* mod$A$.

(b) *If* $\mathrm{fres}(U_A)$ *is semiperfect, then there exists a unique minimal* $\mathrm{fres}(U_A)$-*hull in* mod$A$.

COROLLARY 2.2 [6]. *Let R be a commutative Noetherian complete local ring, A and B finite R-algebras, and U a cotilting B-A-bimodule. Then every finitely generated right A-module has a minimal* $\mathrm{rac}(U_A)$-*approximation and a minimal* $\mathrm{fres}(U_A)$-*hull.*

Proof. By the assumption, $\mathrm{End}_A(M)$ is a semiperfet ring for every finitely generated right A-module M. Then $\mathrm{rac}(U_A)$ and $\mathrm{fres}(U_A)$ are semiperfect. According to [6, Corollary 2.11] and Theorem 2.1, we complete the proof.

LEMMA 2.3. *Let R be a commutative Noetherian local ring, and A an R-algebra. For every finitely generated right A-module X, every right A-module Y, we have the following natural isomorphisms*:

$$\mathrm{Ext}^i_{\hat{A}}(\hat{X}, \hat{Y}) \simeq \mathrm{Ext}^i_A(X,Y) \otimes_R \hat{R} \textit{ for all } i \geq 0.$$

Proof. See [2].

PROPOSITION 2.4. *Let R be a commutative Noetherian local ring, A and B finite R-algebras. If U is a cotilting B-A-bimodule, then $\hat{U}$ is a cotilting $\hat{B}$-$\hat{A}$-bimodule.*

Proof. By Lemma 2.3, U satisfies the conditions (C1), (C3r) and (C4r). Let J

be the radical of A. For all i, we also have the following isomorphism:

$$\operatorname{Ext}^i_{\hat{A}}(\hat{A}/\hat{J},\ \hat{U}) \simeq \operatorname{Ext}^i_A(A/J,U)\otimes_R \hat{R}.$$

Since $\hat{A}$ is a finite $\hat{R}$-algebra and $\hat{J}$ = rad $\hat{A}$, idim$\hat{U}_{\hat{A}}$ is finite by Auslander [8, Proposition 2.7]. Therefore $\hat{U}$ satisfies the condition (C2r). Similarly, $\hat{U}$ satisfies the conditions (C2l), (C3l) and (C4l).

LEMMA 2.5. *Let A be a right Noetherian ring, B a left coherent ring, and U a cotilting B-A-bimodule. Assume that every finitely generated right A-module has a minimal* rac(U_A)*-approximation. For a* rac(U_A)*-approximation of a finitely generated right A-module M*: $0 \to Y \xrightarrow{h} X \to M \to 0$, *the following are equivalent.*

(a) $0 \to Y \xrightarrow{h} X \to M \to 0$ *is a minimal* rac(U_A)*-approximation.*

(b) *There is no non-zero submodule Z of Y such that* $h|_Z$ *is a split monomorphism.*

Proof. (a) ⇒ (b): If there is a non-zero submodule Z of Y such that $h|_Z$ is a split monomorphism. By the proof of [6, Theorem 3.4], we get the following rac(U_A)-approximation:

$$0 \to Y' \to X' \to M \to 0,$$

where $Y = Z\oplus Y'$, $X = Z\oplus X'$ and $Z \in \operatorname{add}U_A = \operatorname{rac}(U_A)\cap\operatorname{fres}(U_A)$. Then it is easy to see that $0 \to Y \to X \to M \to 0$ is not minimal.

(b) ⇒ (a): Let $0 \to Y' \to X' \to M \to 0$ be a minimal rac(U_A)-approximation. By the property of rac(U_A)-approximations, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & Y' & \to & X' & \to & M & \to & 0 \\
 & & \downarrow f & & \downarrow g & & \| & & \\
0 & \to & Y & \to & X & \to & M & \to & 0 \\
 & & \downarrow p & & \downarrow q & & \| & & \\
0 & \to & Y' & \to & X' & \to & M & \to & 0.
\end{array}$$

By the minimality, $q\circ g$ and $p\circ f$ are isomorphisms. If $0 \to Y \to X \to M \to 0$ is not minimal, then p is not a isomorphism. Hence $\mathrm{Ker}p$ is a non-zero submodule of Y such that $h|_{\mathrm{Ker}p}$ is a split monomorphism.

THEOREM 2.6. *Let R be a commutative Noetherian local ring, A and B finite R-algebras, and U a cotilting B-A-bimodule. If B is a semiperfect ring, then every finitely generated right A-module has a minimal* $\mathrm{rac}(U_A)$*-approximation and a minimal* $\mathrm{fres}(U_A)$*-hull.*

Proof. By [6, Proposition 3.2], every finitely generated right A-module M has a $\mathrm{rac}(U_A)$-approximation and a $\mathrm{fres}(U_A)$-hull. Let $\varepsilon : 0 \to Y \xrightarrow{h} X \xrightarrow{k} M \to 0$ be a $\mathrm{rac}(U_A)$-approximation of M. Since Y is Noetherian, there is a maximal submodule Z of Y such that $h|_Z$ is a split monomorphism. By the proof of Lemma 2.5, we may assume that ε satisfies the condition (b) of Lemma 2.5. Applying $\otimes_R \hat{R}$ to ε, we have a $\mathrm{rac}(\hat{U}_{\hat{A}})$-approximation of $\hat{M}$:

$$\hat{\varepsilon} : 0 \to \hat{Y} \xrightarrow{\hat{h}} \hat{X} \xrightarrow{\hat{k}} \hat{M} \to 0.$$

If $\hat{\varepsilon}$ does not satisfy the condition (b) of Lemma 2.5, then there exists a non-zero submodule Z of $\hat{Y}$ such that $\hat{h}|_Z$ is a split monomorphism. Since $B = \mathrm{End}(U_A)$ is a semiperfect ring, U_A is a direct sum $\oplus_{i=1}^n U_i$ where all U_i have local endomorphism rings. Then it is easy to see that $\hat{U}_{\hat{A}}$ is a direct sum $\oplus_{i=1}^n \hat{U}_i$ where all $\hat{U}_i$ have local endomorphism rings. We may assume that Z is isomorphic to $\hat{U}_1$. There exist morphisms f: $\hat{U}_1 \to \hat{Y}$ and g: $\hat{X} \to \hat{U}_1$ such that $g\circ \hat{h}\circ f = 1_{\hat{U}_1}$. In $\mathrm{Hom}_A(U_1,Y)\otimes_R \hat{R} \simeq \mathrm{Hom}_{\hat{A}}(\hat{U}_1, \hat{Y})$, there exist $f_i : U_1 \to Y$ and $r_i \in \hat{R}$ such that a sum $\Sigma_i f_i\otimes r_i$ is corresponding to f. In $\mathrm{Hom}_A(X,U_1)\otimes_R \hat{R} \simeq \mathrm{Hom}_{\hat{A}}(\hat{X}, \hat{U}_1)$, there exist $g_j : X \to U_j$ and $s_j \in \hat{R}$ such that a sum $\Sigma_j g_j\otimes s_j$ is corresponding to g. Therefore, in $\mathrm{End}_A(U_1)\otimes_R \hat{R} \simeq \mathrm{End}_{\hat{A}}(\hat{U}_1)$, a sum $\Sigma_{i,j}\, g_j\circ h\circ f_i \otimes s_j r_i$ is corresponding to $1_{\hat{U}_1}$. Since $\mathrm{End}_A(U_1)$ is a local finite R-algebra, there exists $g_j\circ h\circ f_i$ which is an isomorphism. This contradicts ε satisfies the condition (b) of Lemma 2.5. By Corollary 2.2, Proposition 2.4 and Lemma 2.5, $\hat{\varepsilon}$ is a minimal $\mathrm{rac}(\hat{U}_{\hat{A}})$-approximation. If ε is not a minimal $\mathrm{rac}(U_A)$-approximation, then there exists a non-isomorphism $v : X \to X$ such that $k\circ v = k$. Since $\hat{R}$ is faithfully flat R-module, $\hat{v}$ is a non isomorphism such that $\hat{k}\circ \hat{v} = \hat{k}$.

This contradicts the minimality of $\mathscr{V}$. Simlarly, M has a minimal $\mathrm{fres}(U_A)$-hull.

COROLLARY 2.7. *Let R be a commutative local Cohen-Macaulay ring with a dualizing module ω, A a semiperfect* CM *R-algebra, and* $U := \mathrm{Hom}_R(A, \omega)$. *Then every finitely generated right A-module has a minimal* $\mathrm{CM}(A_A)$*-approximation and a minimal* $\mathrm{fres}(U_A)$*-hull.*

Example. Let k be a field, R a commutative local Cohen-Macaulay k-algebra, and Λ a finite dimensional k-algebra. Then $\Lambda\otimes_k R$ is a semiperfect CM R-algebra if either of the following conditions holds:

(1) $\Lambda/\mathrm{rad}\Lambda$ is isomorphic to $\mathrm{M}_{n_1}(k)\times\mathrm{M}_{n_2}(k)\times\ldots\times\mathrm{M}_{n_r}(k)$,

(2) $R/\mathrm{rad}R$ is isomorphic to k.

Moreover if R has a dualizing R-module, then $\Lambda\otimes_k R$ satisfies the condition of Corollary 2.7.

ACKNOWLEDGMENT

The author would like to thank Professor S. Goto for suggesting the boundedness of derived functors in Theorem 1.2.

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