Derived Categories

Mistuo Hoshino

Contents

§1. Cochain complexes

Throughout this section, $\mathscr{A}$ is an abelian category and $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$. Unless otherwise stated, functors are covariant functors.

Definition 1.1. We denote by $\mathscr{A}^{\mathbb{Z}}$ the category of $\mathbb{Z}$-graded objects in $\mathscr{A}$, i.e., an object of $\mathscr{A}^{\mathbb{Z}}$ is a family $X = \{X^n\}_{n \in \mathbb{Z}}$ with the $X^n \in \mathrm{Ob}(\mathscr{A})$ and a morphism $u : \{X^n\} \to \{Y^n\}$ is a family $u = \{u^n\}_{n \in \mathbb{Z}}$ with the $u^n \in \mathscr{A}(X^n, Y^n)$. We have an autofunctor $T : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$, called a shift functor, such that $(TX)^n = X^{n+1}$ for $X \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ and $(Tu)^n = u^{n+1}$ for $u \in \mathscr{A}^{\mathbb{Z}}(X, Y)$.

Remark 1.1. Each $X \in \mathrm{Ob}(\mathscr{A})$ is considered as a family $\{X^n\}_{n \in \mathbb{Z}}$ such that $X^0 = X$ and $X^n = 0$ for n 0, so that we get a full embedding $\mathscr{A} \to \mathscr{A}^{\mathbb{Z}}$.

Definition 1.2. A cochain complex $X^\bullet = (X, d_X)$ in $\mathscr{A}$ is a pair of $X \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ and $d_X \in \mathscr{A}^{\mathbb{Z}}(X, TX)$ with $Td_X \circ d_X = 0$, where X is called the underlying $\mathbb{Z}$-graded object and d_X is called the differential. A morphism $u : X^\bullet \to Y^\bullet$ of cochain complexes is defined as a morphism $u \in \mathscr{A}^{\mathbb{Z}}(X, Y)$ such that $Tu \circ d_X = d_Y \circ u$. We denote by $C(\mathscr{A})$ the category of cochain complexes in $\mathscr{A}$. We have an autofunctor $T : C(\mathscr{A}) \to C(\mathscr{A})$, called the translation, such that $T X^\bullet = (TX, -Td_X)$ for $X^\bullet = (X, d_X)$. Sometimes, $T^n(X^\bullet)$ is denoted by $X^\bullet[n]$.

Remark 1.2. (1) Each $X \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ is considered as a cohain complex with $d_X = 0$, so that we get a full embedding $\mathscr{A}^{\mathbb{Z}} \to C(\mathscr{A})$. Then $X = \oplus_{n \in \mathbb{Z}} T^{-n}(X^n)$, the direct sum in $C(\mathscr{A})$, for all $X = \{X^n\}_{n \in \mathbb{Z}} \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$.

(2) We set $\overline{X}^\bullet = (X, -d_X)$ for $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. Then $X^\bullet \cong \overline{X}^\bullet$ for all $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$.

Proposition 1.1. $C(\mathscr{A})$ *is an abelian category.*

Proof. For each $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$, Ker u and Cok u are defined by the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\, u^n & \to & X^n & \to & Y^n & \to & \mathrm{Cok}\, u^n & \to & 0 \\
 & & \downarrow & & \downarrow d_X^n & & \downarrow d_Y^n & & \downarrow & & \\
0 & \to & \mathrm{Ker}\, u^{n+1} & \to & X^{n+1} & \to & Y^{n+1} & \to & \mathrm{Cok}\, u^{n+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \;. & &
\end{array}
$$

Then Im $u \cong$ Coim u canonically, so that we may identify Im u with Coim u. Also, the direct sum of two cochain complexes $X^\bullet$, $Y^\bullet$ is defined as follows

$$X^\bullet \oplus Y^\bullet = (X \oplus Y, \begin{bmatrix} d_X & 0 \\ 0 & d_Y \end{bmatrix}).$$

Definition 1.3. We define additive covariant functors $Z^\bullet$, $B^\bullet$, $Z'^\bullet$, $B'^\bullet$ and $H^\bullet : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ as follows:

$$Z^n(X^\bullet) = \operatorname{Ker} d_X^n,$$

$$B^n(X^\bullet) = \operatorname{Im} d_X^{n-1},$$

$$Z'^n(X^\bullet) = \operatorname{Cok} d_X^{n-1},$$

$$B'^n(X^\bullet) = \operatorname{Coim} d_X^n = \operatorname{Im} d_X^n = B^{n+1}(X^\bullet),$$

$$H^n(X^\bullet) = Z^n(X^\bullet)/B^n(X^\bullet).$$

for $X^\bullet \in \operatorname{Ob}(C(\mathscr{A}))$ and $n \in \mathbb{Z}$.

Remark 1.3. (1) $B'^\bullet = T \circ B^\bullet$.
(2) For any $X^\bullet \in \operatorname{Ob}(C(\mathscr{A}))$, d_X admits an epic-monic factorization

$$X^\bullet \to B'^\bullet(X^\bullet) \to T(X^\bullet).$$

Lemma 1.2. (1) $Z^\bullet : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ *is left exact and* $Z'^\bullet : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ *is right exact.*
(2) *We have a commutative diagram of functors with exact rows and columns*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & Z^\bullet & \to & H^\bullet & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & B^\bullet & \to & \mathbf{1}_{C(\mathscr{A})} & \to & Z'^\bullet & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & B'^\bullet & = & B'^\bullet & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

(3) *We have an exact sequence of functors*

$$0 \to H^{\bullet} \to Z'^{\bullet} \to T \circ Z^{\bullet} \to T \circ H^{\bullet} \to 0.$$

Proof. (1) Let $0 \to X^{\bullet} \to Y^{\bullet} \to Z^{\bullet} \to 0$ be an exact sequence in $C(\mathscr{A})$. Applying Snake lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & X^{\bullet} & \to & Y^{\bullet} & \to & Z^{\bullet} & \to & 0 \\ & & {\scriptstyle d_X}\downarrow & & \downarrow{\scriptstyle d_Y} & & \downarrow{\scriptstyle d_Z} & & \\ 0 & \to & T(X^{\bullet}) & \to & T(Y^{\bullet}) & \to & T(Z^{\bullet}) & \to & 0, \end{array}$$

we get an exact sequence

$$0 \to Z^{\bullet}(X^{\bullet}) \to Z^{\bullet}(Y^{\bullet}) \to Z^{\bullet}(Z^{\bullet}) \to Z'^{\bullet}(T(X^{\bullet})) \to Z'^{\bullet}(T(Y^{\bullet})) \to Z'^{\bullet}(T(Z^{\bullet})) \to 0.$$

(2) Straightforward.

(3) In the diagram of the part (2), since $B'^{\bullet} = T \circ B^{\bullet}$, we can splice the top row, shifted by one, with the right end column to get a desired exact sequence.

Proposition 1.3. *Let* $0 \to X^{\bullet} \to Y^{\bullet} \to Z^{\bullet} \to 0$ *be an exact sequence in* $C(\mathscr{A})$. *Then we have a long exact sequence in* $\mathscr{A}$

$$\cdots \to H^n(X^{\bullet}) \to H^n(Y^{\bullet}) \to H^n(Z^{\bullet}) \xrightarrow{\omega^n} H^{n+1}(X^{\bullet}) \to \cdots .$$

Proof. By Lemma 1.2(1) we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & Z'^{\bullet}(X^{\bullet}) & \to & Z'^{\bullet}(Y^{\bullet}) & \to & Z'^{\bullet}(Z^{\bullet}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T(Z^{\bullet}(X^{\bullet})) & \to & T(Z^{\bullet}(Y^{\bullet})) & \to & T(Z^{\bullet}(Z^{\bullet})), & & \end{array}$$

to which we apply Snake lemma. Then by Lemma 1.2(3) we get an exact sequence in $\mathscr{A}^{\mathbb{Z}}$

$$H^{\bullet}(X^{\bullet}) \to H^{\bullet}(Y^{\bullet}) \to H^{\bullet}(Z^{\bullet}) \xrightarrow{\omega} T(H^{\bullet}(X^{\bullet})) \to T(H^{\bullet}(Y^{\bullet})) \to T(H^{\bullet}(Z^{\bullet})).$$

Definition 1.4. Let $\mathscr{B}$ be another abelian category. Then every additive covariant (resp. contravariant) functor $F : \mathscr{A} \to \mathscr{B}$ can be extended to an additive covariant (resp. contravariant) functor $F : C(\mathscr{A}) \to C(\mathscr{B})$ as follows: if $F : \mathscr{A} \to \mathscr{B}$ is covariant, then $F : C(\mathscr{A}) \to C(\mathscr{B})$ associates with each $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ a cochain complex $F X^\bullet$ such that $(F X^\bullet)^n = F(X^n)$ and $d^n_{FX} = F(d^n_X)$ for all $n \in \mathbb{Z}$; and if $F : \mathscr{A} \to \mathscr{B}$ is contravariant, then $F : C(\mathscr{A}) \to C(\mathscr{B})$ associates with each $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ a cochain complex $F X^\bullet$ such that $(F X^\bullet)^n = F(X^{-n})$ and $d^n_{FX} = F(d_X^{-(n+1)})$ for all $n \in \mathbb{Z}$.

Proposition 1.4. *Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. Then for the extended functor $F : C(\mathscr{A}) \to C(\mathscr{B})$ the following hold.*

(1) *If F is covariant, then $F \circ T = T \circ F$.*

(2) *If F is contravariant, then $F \circ T = T^{-1} \circ F$.*

Proof. Straightforward.

Proposition 1.5. *Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \to \mathscr{B}$ an exact functor. Then for the extended functor $F : C(\mathscr{A}) \to C(\mathscr{B})$ the following hold.*

(1) *If F is covariant, then $F \circ H^n \cong H^n \circ F$ for all $n \in \mathbb{Z}$.*

(2) *If F is contravariant, then $F \circ H^n \cong H^{-n} \circ F$ for all $n \in \mathbb{Z}$.*

Proof. Straightforward

Definition 1.5. We denote by $U : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ the underlying functor, i.e., U associates with each complex $X^\bullet = (X, d_X)$ its underlying $\mathbb{Z}$-graded object X.

Proposition 1.6. *Let $U : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ denote the underlying functor. Then the following hold.*

(1) *U is exact and has both a right adjoint $S : \mathscr{A}^{\mathbb{Z}} \to C(\mathscr{A})$ which associates with each $\mathbb{Z}$-graded object X a complex*

$$S(X) = (TX \oplus X, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$$

and a left adjoint $T^{-1} \circ S : \mathscr{A}^{\mathbb{Z}} \to C(\mathscr{A})$.

(2) *We have an exact sequence of functors*

$$0 \to \mathbf{1}_{C(\mathscr{A})} \xrightarrow{\mu} S \circ U \xrightarrow{\varepsilon} T \to 0,$$

where $\mu = {}^{t}[d \quad 1]$, $\varepsilon = [1 \quad -d]$.

Proof. (1) It is obvious that U is exact. For any $X^\bullet \in \mathrm{Ob}(C(\mathcal{A}))$ and $Y \in \mathrm{Ob}(\mathcal{A}^{\mathbb{Z}})$, we have natural isomorphisms

$$\mathcal{A}^{\mathbb{Z}}(U(X^\bullet), Y) \xrightarrow{\sim} C(\mathcal{A})(X^\bullet, S(Y)),\ u \mapsto \begin{bmatrix} Tu \circ d_X \\ u \end{bmatrix},$$

$$\mathcal{A}^{\mathbb{Z}}(Y, U(X^\bullet)) \xrightarrow{\sim} C(\mathcal{A})(T^{-1}S(Y), X^\bullet),\ u \mapsto [u \quad T^{-1}(d_X \circ u)].$$

(2) Straightforward.

Remark 1.4. (1) $T \circ S \cong S \circ T$.

(2) Let $X^\bullet \in \mathrm{Ob}(C(\mathcal{A}))$. If the canonical exact sequence

$$0 \to X^\bullet \xrightarrow{\mu_X} S(U(X^\bullet)) \xrightarrow{\varepsilon_X} TX^\bullet \to 0$$

splits, then $X^\bullet \cong S(Z^\bullet(X^\bullet))$.

(3) An object $I \in \mathrm{Ob}(\mathcal{A}^{\mathbb{Z}})$ is injective if and only if so is $S(I) \in \mathrm{Ob}(C(\mathcal{A}))$.

(4) An object $X^\bullet \in \mathrm{Ob}(C(\mathcal{A}))$ is injective if and only if $X^\bullet \cong S(I)$ with $I \in \mathrm{Ob}(\mathcal{A}^{\mathbb{Z}})$ injective.

Definition 1.6. An abelian category $\mathcal{A}$ is said to have enough injectives (resp. projectives) if for each $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $X \to I$ with $I \in \mathcal{I}$ (resp. an epimorphism $P \to X$ with $P \in \mathcal{P}$).

Lemma 1.7. *Let* $\mathcal{L}$ *be a subcollection of* $\mathrm{Ob}(\mathcal{A})$ *containing zero objects and closed under finite direct sums and assume for any* $X \in \mathrm{Ob}(\mathcal{A})$ *there exists a monomorphism* $X \to I$ *in* $\mathcal{A}$ *with* $I \in \mathcal{L}$. *Then for any* $X^\bullet \in \mathrm{Ob}(C(\mathcal{A}))$ *there exists a monomorphism* $X^\bullet \to I^\bullet$ *in* $C(\mathcal{A})$ *with the* $I^n \in \mathcal{L}$. *In particular, if* $\mathcal{A}$ *has enough injectives, so does* $C(\mathcal{A})$.

Proof. For each $n \in \mathbb{Z}$, we have a monomorphism $u^n : X^n \to I^n$ in $\mathcal{A}$ with $I^n \in \mathcal{L}$. Thus we get a monomorphism $u = \{u^n\} : U(X^\bullet) \to I = \{I^n\}$ in $\mathcal{A}^{\mathbb{Z}}$. Then, since $S : \mathcal{A}^{\mathbb{Z}} \to C(\mathcal{A})$ is exact, we get a monomorphism $S(u) : S(U(X^\bullet)) \to S(I)$ in $C(\mathcal{A})$. Thus by Proposition 1.6(2) we get a monomorphism $X^\bullet \to S(I)$ in $C(\mathcal{A})$. Since by Proposition 1.6(1) $S : \mathcal{A}^{\mathbb{Z}} \to C(\mathcal{A})$ takes injective objects into injective objects, if the I^n are injective, so is $S(I)$.

Definition 1.7. A complex $X^\bullet$ is called bounded below if $X^n = 0$ for $n \ll 0$, bounded above if $X^n = 0$ for $n \gg 0$ and bounded if $X^n = 0$ for $n \ll 0$ and $n \gg 0$. We denote by $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^{\mathrm{b}}(\mathcal{A})$ the full subcategory of $C(\mathcal{A})$ consisting of bounded below complexes, bounded above complexes and bounded complexes, respectively.

Also, a complex $X^\bullet$ is said to have a bounded below cohomology if $H^n(X^\bullet) = 0$ for $n \ll 0$, to have a bounded above cohomology if $H^n(X^\bullet) = 0$ for $n \gg 0$, and to have a bounded cohomology if $H^n(X^\bullet) = 0$ for $n \ll 0$ and for $n \gg 0$. For $* = +$ or $-$, we denote by $C^{*,\,\mathrm{b}}(\mathscr{A})$ the full subcategory of $C^*(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(C^*(\mathscr{A}))$ with bounded cohomology.

Remark 1.5. For $* = +, -$ or b, $C^*(\mathscr{A})$ is an abelian exact full subcategory of $C(\mathscr{A})$.

Definition 1.8. Let $\mathscr{L}$ be a subcollection of $\mathrm{Ob}(\mathscr{A})$. For $* = +, -$, b or nothing, we denote by $C^*(\mathscr{L})$ the full subcategory of $C^*(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(C^*(\mathscr{A}))$ with $X^n \in \mathscr{L}$ for all $n \in \mathbb{Z}$.

Remark 1.6. Let $\mathscr{L}$ be a subcollection of $\mathrm{Ob}(\mathscr{A})$ containing zero objects and closed under finite direct sums. Then, for $* = +, -$, b or nothing, $C^*(\mathscr{L})$ is an additive full subcategory of $C^*(\mathscr{A})$.

Definition 1.9. A right resolution of $X \in \mathrm{Ob}(\mathscr{A})$ is a morphism $\mu : X \to I^\bullet$ in $C(\mathscr{A})$ such that $H^0(\mu) : X \to H^0(I^\bullet)$ is an isomorphism, $I^n = 0$ for all $n < 0$ and $H^n(I^\bullet) = 0$ for all $n > 0$, i.e., we have an exact sequence $0 \to X \xrightarrow{\mu} I^0 \to I^1 \to \cdots$.

A right resolution $\mu : X \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(C(\mathscr{I}))$ is called an injective resolution. Let $f \in \mathscr{A}(X, Y)$ and $\mu_X : X \to I_X^\bullet$, $\mu_Y : Y \to I_Y^\bullet$ be right resolutions of X and Y, respectively. Then a morphism $\hat{f} \in C(\mathscr{A})(I_X^\bullet, I_Y^\bullet)$ with $\hat{f} \circ \mu_X = \mu_Y \circ f$ is said to be lying over f.

Lemma 1.8. *Let $I^\bullet \in \mathrm{Ob}(C(\mathscr{I}))$ with $I^n = 0$ for $n < 0$ and $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ with $H^n(X^\bullet) = 0$ for $n > 0$. Then for any $f : H^0(X^\bullet) \to H^0(I^\bullet)$ the following hold.*

(1) *There exists $\hat{f} \in C(\mathscr{A})(X^\bullet, I^\bullet)$ such that $H^0(\hat{f}) = f$.*

(2) *In case $f = 0$, for any $\hat{f} \in C(\mathscr{A})(X^\bullet, I^\bullet)$ with $H^0(\hat{f}) = f$ there exists $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, I^\bullet)$ such that $\hat{f} = h \circ d_X + T^{-1}(d_I \circ h)$.*

Proof. (1) Put $\hat{f}^n = 0$ for $n < 0$. By the injectivity of $Z'^0(I^\bullet) = I^0$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^0(X^\bullet) & \to & Z'^0(X^\bullet) & \to & Z^1(X^\bullet) & \to & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle z^1} & & \\ 0 & \to & H^0(I^\bullet) & \to & I^0 & \to & Z^1(I^\bullet). & & \end{array}$$

Let $\pi : X^0 \to Z'^0(X^\bullet)$ be the canonical epimorphism and put $\hat{f}^0 = g \circ \pi$. Then $\hat{f}^0 \circ d_X^{-1} = 0$.

Let $n \geq 1$ and assume that $z^n : Z^n(X^\bullet) \to Z^n(I^\bullet)$ has been constructed. Then, by the injectivity of I^n, there exist $\hat{f}^n : X^n \to I^n$ and $z^{n+1} : Z^{n+1}(X^\bullet) \to Z^{n+1}(I^\bullet)$ which make the following diagram with exact rows commute

$$\begin{array}{ccccccccc} 0 & \to & Z^n(X^\bullet) & \to & X^n & \to & Z^{n+1}(X^\bullet) & \to & 0 \\ & & {\scriptstyle z^n}\downarrow & & \downarrow{\scriptstyle \hat{f}^n} & & \downarrow{\scriptstyle z^{n+1}} & & \\ 0 & \to & Z^n(I^\bullet) & \to & I^n & \to & Z^{n+1}(I^\bullet). & & \end{array}$$

Thus by induction we get a desired morphism $\hat{f} \in C(\mathscr{A})(X^\bullet, I^\bullet)$.

(2) Put $h^n = 0$ for $n < 0$. Note that $Z^0(\hat{f}) = 0$. Let $\mu : Z^0(X^\bullet) \to X^0$ be the inclusion. Since $\hat{f}^0 \circ \mu = 0$, there exists $h^1 : X^1 \to I_Y^0$ such that $\hat{f}^0 = h^0 \circ d_X^0 + h^{-1} \circ d_I^{-1}$. It suffices to prove the following.

Claim: Let $n \geq 0$ and assume that, for $-1 \leq i \leq n$, the $h^i : X^{i+1} \to I^i$ have been constructed to satisfy $\hat{f}^i = h^i \circ d_X^i + d_I^{i-1} \circ h^{i-1}$ for all $0 \leq i \leq n$. Then there exists $h^{n+1} : X^{n+2} \to I^{n+1}$ such that $\hat{f}^{n+1} = d_I^n \circ h^n + h^{n+1} \circ d_X^{n+1}$.

Proof. We have $(\hat{f}^{n+1} - d_I^n \circ h^n) \circ d_X^n = 0$, so that $(\hat{f}^{n+1} - d_I^n \circ h^n)$ factors through d_X^{n+1}.

Definition 1.10. A left resolution of $X \in \mathrm{Ob}(\mathscr{A})$ is a morphism $\varepsilon : P^\bullet \to X$ in $C(\mathscr{A})$ such that $H^0(\varepsilon) : H^0(P^\bullet) \to X$ is an isomorphism, $P^n = 0$ for all $n > 0$ and $H^n(P^\bullet) = 0$ for all $n < 0$, i.e., we have an exact sequence $\cdots \to P^{-1} \to P^0 \xrightarrow{\varepsilon} X \to 0$.

A left resolution $\varepsilon : P^\bullet \to X$ with $P^\bullet \in \mathrm{Ob}(C(\mathscr{P}))$ is called a projective resolution. Let $f \in \mathscr{A}(X, Y)$ and $\varepsilon_X : P_X^\bullet \to X$, $\varepsilon_Y : P_Y^\bullet \to Y$ be left resolutions of X and Y, respectively. Then a morphism $\hat{f} \in C(\mathscr{A})(P_X^\bullet, P_Y^\bullet)$ with $\varepsilon_Y \circ \hat{f} = f \circ \varepsilon_X$ is said to be lying over f.

Lemma 1.9 (Dual of Lemma 1.7). *Let $\mathscr{L}$ be a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *such that for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists an epimorphism $P \to X$ in $\mathscr{A}$ with $P \in \mathscr{L}$. Then for any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ there exists an epimorphism $P^\bullet \to X^\bullet$ in $C(\mathscr{A})$ with the $P^n \in \mathscr{L}$. In particular, if $\mathscr{A}$ has enough projectives, so does $C(\mathscr{A})$.*

Lemma 1.10 (Dual of Lemma 1.8). *Let $P^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ with the $P^n \in \mathscr{P}$ and $P^n = 0$ for $n > 0$, and let $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ with $H^n(X^\bullet) = 0$ for $n < 0$. Then for any $f : H^0(P^\bullet) \to H^0(X^\bullet)$ the following hold.*

(1) *There exists $\hat{f} \in C(\mathscr{A})(P^\bullet, X^\bullet)$ such that $H^0(\hat{f}) = f$.*

(2) *In case $f = 0$, for any $\hat{f} \in C(\mathscr{A})(P^\bullet, X^\bullet)$ with $H^0(\hat{f}) = f$ there exists $h \in \mathscr{A}^{\mathbb{Z}}(TP^\bullet, X^\bullet)$ such that $\hat{f} = h \circ d_P + T^{-1}(d_X \circ h)$.*

Definition 1.11. Let $\mathscr{C}$ be a category and Λ a set. We may consider Λ as a discrete category. Namely, Λ is considered as a category such that $\mathrm{Ob}(\Lambda) = \Lambda$ and there is no other morphism than identity morphisms. Note that $\Lambda^{\mathrm{op}} = \Lambda$. We denote by $\mathscr{C}^{\Lambda}$ the functor category $[\Lambda, \mathscr{C}]$. Then an object of $\mathscr{C}^{\Lambda}$ is just a family of objects $\{X_\lambda\}_{\lambda \in \Lambda}$ in $\mathscr{C}$. We have a functor $P : \mathscr{C} \to \mathscr{C}^{\Lambda}$, called the constant functor, which associates with each $X \in \mathrm{Ob}(\mathscr{C})$ a family of objects $\{X_\lambda\}_{\lambda \in \Lambda}$ such that $X_\lambda = X$ for all $\lambda \in \Lambda$.

A product of $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^{\Lambda})$ is a terminal object in the following category: an object is a morphism in $\mathscr{C}^{\Lambda}$ of the form $f \in \mathscr{C}^{\Lambda}(PX, \{X_\lambda\})$ with $X \in \mathrm{Ob}(\mathscr{C})$, i.e., a pair $(X, \{f_\lambda\})$ of $X \in \mathrm{Ob}(\mathscr{C})$ and a family of morphisms $f_\lambda \in \mathscr{C}(X, X_\lambda)$; a morphism $h : (X, \{f_\lambda\}) \to (Y, \{g_\lambda\})$ is a morphism $h \in \mathscr{C}(X, Y)$ with $f_\lambda = g_\lambda \circ h$ for all $\lambda \in \Lambda$. If $\{X_\lambda\}$ has a product $(X, \{p_\lambda\})$, then the morphisms $p_\lambda : X \to X_\lambda$ are called projections. In case $\mathscr{C}$ is an additive category, a product is usually called a direct product.

Remark 1.7. Let $\mathscr{C}$ be a category and Λ a set. Then the following hold.

(1) A pair $(X, \{p_\lambda\})$ is a product of $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^{\Lambda})$ if and only if the mapping

$$\mathscr{C}(Y, X) \to \prod \mathscr{C}(Y, X_\lambda),\ f \mapsto (p_\lambda \circ f)$$

is a bijection for all $Y \in \mathrm{Ob}(\mathscr{C})$.

(2) Assume every $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^{\Lambda})$ has a product $(\prod X_\lambda, \{p_\lambda\})$. Then $\prod : \mathscr{C}^{\Lambda} \to \mathscr{C}$ is a functor and is a right adjoint of the constant functor $P : \mathscr{C} \to \mathscr{C}^{\Lambda}$. Furthermore, the morphisms $p = \{p_\lambda\} : P(\prod X_\lambda) \to \{X_\lambda\}$ give rise to the counit.

(3) Assume the constant functor $P : \mathscr{C} \to \mathscr{C}^{\Lambda}$ has a right adjoint $\prod : \mathscr{C}^{\Lambda} \to \mathscr{C}$ and denote by $p : P \circ \prod \to \mathbf{1}_{\mathscr{C}^{\Lambda}}$ the counit. Then every $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^{\Lambda})$ has a product $(\prod X_\lambda, \{p_\lambda\})$.

Definition 1.12. An abelian category $\mathscr{A}$ is said to satisfy the condition Ab3* if arbitrary direct products exist in $\mathscr{A}$, and to satisfiy the condition Ab4* if arbitrary direct products exist in $\mathscr{A}$ and for any set Λ the functor $\prod : \mathscr{A}^{\Lambda} \to \mathscr{A}$ is exact.

Remark 1.8. If $\mathscr{A}$ satisfies the condition Ab3*, then for any set Λ the functor $\prod : \mathscr{A}^{\Lambda} \to \mathscr{A}$ is a right adjoint of the constant functor $\mathscr{A} \to \mathscr{A}^{\Lambda}$ and thus left exact.

Proposition 1.11. (1) *If $\mathscr{A}$ satisfies the condition Ab3*, then so does $C(\mathscr{A})$.*

(2) *If $\mathscr{A}$ satisfies the condition Ab4*, then so does $C(\mathscr{A})$ and $H^{\bullet} : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ commutes with direct products.*

Proof. (1) Let $\{X^{\bullet}_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of complexes in $C(\mathscr{A})$. Then we have a complex $\prod X^\bullet_\lambda$ such that

$$(\prod X^\bullet_\lambda)^n = \prod X^n_\lambda, \quad d^n_{\prod X^\bullet_\lambda} = \prod d^n_{X^\bullet_\lambda}$$

for all $n \in \mathbb{Z}$. Also, for each $\mu \in \Lambda$, we have a homomorphism $p_\mu : \prod X^\bullet_\lambda \to X^\bullet_\mu$ in $C(\mathscr{A})$ such that $p^n_\mu : \prod X^n_\lambda \to X^n_\mu$ is a projection for all $n \in \mathbb{Z}$. It is easy to see that for any $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ the canonical homomorphism

$$C(\mathscr{A})(Y^\bullet, \prod X^\bullet_\lambda) \to \prod C(\mathscr{A})(Y^\bullet, X^\bullet_\lambda), u \mapsto (p_\lambda \circ u)$$

is an isomorphism. Thus $(\prod X^\bullet_\lambda, \{p_\lambda\})$ is a direct product of $\{X^\bullet_\lambda\}_{\lambda \in \Lambda}$.

(2) Straightforward.

Definition 1.13. Let $\mathscr{C}$ be a category and Λ a set. Denote by $P : \mathscr{C} \to \mathscr{C}^\Lambda$ the constant functor. A coproduct of $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ is an initial object in the following category: an object is a morphism in $\mathscr{C}^\Lambda$ of the form $f \in \mathscr{C}^\Lambda(\{X_\lambda\}, PX)$ with $X \in \mathrm{Ob}(\mathscr{C})$, i.e., a pair $(\{f_\lambda\}, X)$ of $X \in \mathrm{Ob}(\mathscr{C})$ and a family of morphisms $f_\lambda \in \mathscr{C}(X_\lambda, X)$; a morphism $h : (\{f_\lambda\}, X) \to (\{g_\lambda\}, Y)$ is a morphism $h \in \mathscr{C}(X, Y)$ with $g_\lambda = h \circ f_\lambda$ for all $\lambda \in \Lambda$. If $\{X_\lambda\}$ has a coproduct $(\{i_\lambda\}, X)$, then the morphisms $i_\lambda : X_\lambda \to X$ are called Injections. In case $\mathscr{C}$ is an additive category, a coproduct is usually called a direct sum.

Remark 1.9. Let $\mathscr{C}$ be a category and Λ a set. Then the following hold.

(1) A pair $(\{i_\lambda\}, X)$ is a coproduct of $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ if and only if the mapping

$$\mathscr{C}(X, Y) \to \prod \mathscr{C}(X_\lambda, Y), f \mapsto (f \circ i_\lambda)$$

is a bijection for all $Y \in \mathrm{Ob}(\mathscr{C})$.

(2) Assume every $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a coproduct $(\{i_\lambda\}, \oplus X_\lambda)$. Then $\oplus : \mathscr{C}^\Lambda \to \mathscr{C}$ is a functor and is a left adjoint of the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$. Furthermore, the morphisms $i = \{i_\lambda\} : \{X_\lambda\} \to P(\oplus X_\lambda)$ give rise to the unit.

(3) Assume the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$ has a left adjoint $\oplus : \mathscr{C}^\Lambda \to \mathscr{C}$ and denote by $i : \mathbf{1}_{\mathscr{C}^\Lambda} \to P \circ \oplus$ the unit. Then every $\{X_\lambda\} \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a coproduct $(\{i_\lambda\}, \oplus X_\lambda)$.

Definition 1.14. An abelian category $\mathscr{A}$ is said to satisfy the condition Ab3 if arbitrary direct sums exist in $\mathscr{A}$, and to satisfiy the condition Ab4 if arbitrary direct sums exist in $\mathscr{A}$ and for any set Λ the functor $\oplus : \mathscr{A}^\Lambda \to \mathscr{A}$ is exact.

Remark 1.10. If $\mathscr{A}$ satisfies the condition Ab3, then for any set Λ the functor $\oplus : \mathscr{A}^\Lambda \to \mathscr{A}$

is a left adjoint of the constant functor $\mathscr{A} \to \mathscr{A}^{\Lambda}$ and thus right exact.

Proposition 1.12 (Dual of Proposition 1.11). (1) *If $\mathscr{A}$ satisfies the condition Ab3, then so does $C(\mathscr{A})$.*

(2) *If $\mathscr{A}$ satisfies the condition Ab4, then so does $C(\mathscr{A})$ and $H^{\bullet} : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ commutes with direct sums.*

§2. Mapping cones

Throughout this section, $\mathscr{A}$ is an abelian category and $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$. Unless otherwise stated, functors are covariant functors.

Definition 2.1. The mapping cone of $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ is a cochain complex of the form

$$C(u) = (TX \oplus Y, \begin{bmatrix} d_{TX} & 0 \\ Tu & d_Y \end{bmatrix}).$$

Remark 2.1. (1) $T^n(C(u)) \cong C((-1)^n T^n u)$ for all $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ and $n \in \mathbb{Z}$.

(2) Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. Then $F(C(u)) \cong C(Fu)$ for all $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$. In case F is contravariant, $F(C(u)) \cong C(T^{-1}(Fu))$.

(3) $X^\bullet \oplus Y^\bullet \cong C(0_{T^{-1}X, Y})$ for all $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$.

(4) For any $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, $U X^\bullet \cong Z^\bullet(C(\mathrm{id}_X))$ and we have an isomorphism

$$\begin{bmatrix} 1 & d_X \\ 0 & 1 \end{bmatrix} : C(\mathrm{id}_X) \xrightarrow{\sim} S(U(X^\bullet)).$$

Proposition 2.1. *For any $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ we have an exact sequence in $C(\mathscr{A})$*

$$0 \to Y^\bullet \xrightarrow{\mu} C(u) \xrightarrow{\varepsilon} T X^\bullet \to 0,$$

where $\mu = {}^t[0 \ \ 1]$ *and* $\varepsilon = [1 \ \ 0]$.

Proof. Straightforward.

Proposition 2.2. *For any $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & X^\bullet & \xrightarrow{\mu} & C(\mathrm{id}_X) & \xrightarrow{\varepsilon} & T X^\bullet & \to & 0 \\ & & {\scriptstyle u}\downarrow & \mathrm{PO} & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \to & Y^\bullet & \xrightarrow{\mu} & C(u) & \xrightarrow{\varepsilon} & T X^\bullet & \to & 0 \\ & & \| & & {\scriptstyle -\psi}\downarrow & \mathrm{PB} & \downarrow{\scriptstyle -Tu} & & \\ 0 & \to & Y^\bullet & \xrightarrow{-\mu} & C(\mathrm{id}_Y) & \xrightarrow{\varepsilon} & T Y^\bullet & \to & 0, \end{array}$$

where $\mu = {}^{t}[0\ \ 1]$, $\varepsilon = [1\ \ 0]$, $\phi = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ and $\psi = \begin{bmatrix} Tu & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. Straightforward.

Definition 2.2. A complex $X^{\bullet} \in \mathrm{Ob}(C(\mathscr{A}))$ is caled acyclic if $H^{\bullet}(X^{\bullet}) = 0$.

Remark 2.2. Let $\mu : X \to I^{\bullet}$ be a right resolution of $X \in \mathrm{Ob}(\mathscr{A})$. Then $C(\mu)$ is an acyclic complex

$$\cdots \to 0 \to X \xrightarrow{\mu} I^{0} \to I^{1} \to \cdots .$$

Proposition 2.3. $C(\mathrm{id}_X)$ *is acyclic for all* $X^{\bullet} \in \mathrm{Ob}(C(\mathscr{A}))$.

Proof. Let $n \in \mathbb{Z}$. Let ${}^{t}[u\ \ v] : Y \to X^{n+1} \oplus X^{n}$ be a morphism in $\mathscr{A}$ with

$$\begin{bmatrix} -d_X^{n+1} & 0 \\ 1 & d_X^{n} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Then $u = -d_X^{n} \circ v$ and we have

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -d_X^{n} & 0 \\ 1 & d_X^{n-1} \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

Thus $H^{n}(C(\mathrm{id}_X)) = 0$.

Proposition 2.4. *For any* $u \in C(\mathscr{A})(X^{\bullet}, Y^{\bullet})$ *we have a long exact sequence*

$$\cdots \to H^{n}(X^{\bullet}) \to H^{n}(Y^{\bullet}) \to H^{n}(C(u)) \to H^{n+1}(X^{\bullet}) \to \cdots .$$

Proof. By Proposition 2.2 we have an exact sequence of the form

$$0 \to X^{\bullet} \to Y^{\bullet} \oplus C(\mathrm{id}_X) \to C(u) \to 0.$$

Since by Proposition 2.3 $H^{n}(C(\mathrm{id}_X)) = 0$ for all $n \in \mathbb{Z}$, by Proposition 1.3 the assertion follows.

Proposition 2.5. *For any exact sequence* $0 \to X^{\bullet} \xrightarrow{u} Y^{\bullet} \xrightarrow{v} Z^{\bullet} \to 0$ *in* $C(\mathscr{A})$ *the following hold*.

(1) *We have the following exact diagram in* $C(\mathscr{A})$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X^\bullet & \xrightarrow{\mu} & C(\mathrm{id}_X) & \xrightarrow{\varepsilon} & TX^\bullet & \to & 0 \\
 & & u\downarrow & \mathrm{PO} & \downarrow\phi & & \| & & \\
0 & \to & Y^\bullet & \xrightarrow{\mu} & C(u) & \xrightarrow{\varepsilon} & TX^\bullet & \to & 0 \\
 & & v\downarrow & & \downarrow\pi & & & & \\
 & & Z^\bullet & = & Z^\bullet & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

where $\mu = {}^{\mathrm{t}}[0\ \ 1]$, $\varepsilon = [1\ \ 0]$, $\phi = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ *and* $\pi = [0\ \ v]$.

(2) $H^\bullet(\pi) : H^\bullet(C(u)) \to H^\bullet(Z^\bullet)$ *is an isomorphism.*

(3) *The composite*

$$\omega = H^\bullet(\varepsilon) \circ H^\bullet(\pi)^{-1} : H^\bullet(Z^\bullet) \to H^\bullet(TX^\bullet) = T(H^\bullet(X^\bullet))$$

gives rise to a connecting morphism of a long exact sequence

$$\cdots \to H^n(Y^\bullet) \to H^n(Z^\bullet) \xrightarrow{\omega^n} H^{n+1}(X^\bullet) \to H^{n+1}(Y^\bullet) \to \cdots .$$

Proof. (1) According to Proposition 2.2, it only remains to check that π is a morphism in $C(\mathscr{A})$. We have

$$\begin{aligned} T\pi \circ d_{C(u)} &= [0\ \ Tv]\begin{bmatrix} d_{TX} & 0 \\ Tu & d_Y \end{bmatrix} \\ &= [0\ \ Tv \circ d_Y] \\ &= [0\ \ d_Z \circ v] \\ &= d_Z \circ \pi. \end{aligned}$$

(2) By the part (1) we have an exact sequence $0 \to C(\mathrm{id}_X) \to C(u) \xrightarrow{\pi} Z^\bullet \to 0$. Thus by Propositions 1.3 and 2.3 $H^n(\pi)$ is an isomorphism for all $n \in \mathbb{Z}$.

(3) Let $n \in \mathbb{Z}$. By Propositions 2.2, 2.3 and 1.3 we have an exact sequence

$$H^n(Y^\bullet) \xrightarrow{H^n(\mu)} H^n(C(u)) \xrightarrow{H^n(\varepsilon)} H^{n+1}(X^\bullet) \xrightarrow{H^{n+1}(u)} H^{n+1}(Y^\bullet).$$

Since $H^n(\mu) = H^n(\pi)^{-1} \circ H^n(v)$ and $\omega^n = H^n(\varepsilon) \circ H^n(\pi)^{-1}$, the diagram

$$\begin{array}{ccccccc} H^n(Y^\bullet) & \xrightarrow{H^n(v)} & H^n(Z^\bullet) & \xrightarrow{\omega^n} & H^{n+1}(X^\bullet) & \xrightarrow{H^{n+1}(u)} & H^{n+1}(Y^\bullet) \\ \| & & \downarrow H^n(\pi)^{-1} & & \| & & \| \\ H^n(Y^\bullet) & \xrightarrow{H^n(\mu)} & H^n(C(u)) & \xrightarrow{H^n(\varepsilon)} & H^{n+1}(X^\bullet) & \xrightarrow{H^{n+1}(u)} & H^{n+1}(Y^\bullet) \end{array}$$

commutes and the top row is exact.

Lemma 2.6. *Let $0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$ be an exact sequence in $C(\mathscr{A})$ which splits as an exact sequence in $\mathscr{A}^{\mathbb{Z}}$. Then there exist $w : T^{-1} Z^\bullet \to X^\bullet$ and $\phi : Y^\bullet \to C(w)$ such that the following diagram in $C(\mathscr{A})$ commutes*

$$\begin{array}{ccccccccc} 0 & \to & X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{v} & Z^\bullet & \to & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \to & X^\bullet & \xrightarrow{\mu} & C(w) & \xrightarrow{\varepsilon} & Z^\bullet & \to & 0, \end{array}$$

where $\mu = {}^{t}[0 \ \ 1]$, $\varepsilon = [1 \ \ 0]$.

Proof. Let $f \in \mathscr{A}^{\mathbb{Z}}(Y^\bullet, X^\bullet)$ with $fu = \mathrm{id}_X$. Then $g = Tf \circ d_Y - d_X \circ f \in C(\mathscr{A})(Y^\bullet, TX^\bullet)$ and $gu = 0$. Thus there exists $w \in C(\mathscr{A})(T^{-1} Z^\bullet, X^\bullet)$ such that $g = Tw \circ v$. Finally, it is easy to see that $\phi = {}^{t}[v \ \ f] \in C(\mathscr{A})(Y^\bullet, C(w))$.

Proposition 2.7. *Let $0 \to X \xrightarrow{\mu} Y \xrightarrow{\varepsilon} Z \to 0$ be an exact sequence in $\mathscr{A}$. Let $\mu_X : X \to I_X^\bullet$ be an injective resolution of X and $\mu_Z : Z \to I_Z^\bullet$ a right resolution of Z. Then there exists a right resolution $\mu_Y : Y \to I_Y^\bullet$ of Y such that we have a commutative diagram in $C(\mathscr{A})$ with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{\mu} & Y & \xrightarrow{\varepsilon} & Z & \to & 0 \\ & & \mu_X \downarrow & & \downarrow \mu_Y & & \downarrow \mu_Z & & \\ 0 & \to & I_X^\bullet & \xrightarrow{\hat{\mu}} & I_Y^\bullet & \xrightarrow{\hat{\varepsilon}} & I_Z^\bullet & \to & 0. \end{array}$$

Furthermore, we may assume that $I_Y^\bullet = C(\delta)$ with $\delta \in C(\mathscr{A})(T^{-1} I_Z^\bullet, I_X^\bullet)$, $\mu_Y = {}^{t}[\mu_Z \circ \varepsilon \quad \delta^{-1}]$ with $\delta^{-1} \in \mathscr{A}(Y, I_X^0)$, $\hat{\mu} = {}^{t}[0 \ \ 1]$ and $\hat{\varepsilon} = [1 \ \ 0]$.

Proof. By Lemma 1.8 we have a commutative diagram in $\mathscr{A}$ with exact rows

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu} & Y & \xrightarrow{-\mu_Z \circ \varepsilon} & I_Z^0 & \xrightarrow{-d_{I_Z}^0} & I_Z^1 & \xrightarrow{-d_{I_Z}^1} & \cdots \\
 & & \| & & \downarrow \delta^{-1} & & \downarrow \delta^0 & & \downarrow \delta^1 & & \\
0 & \longrightarrow & X & \xrightarrow{\mu_X} & I_X^0 & \xrightarrow{d_{I_X}^0} & I_X^1 & \xrightarrow{d_{I_X}^1} & I_X^2 & \xrightarrow{d_{I_X}^2} & \cdots .
\end{array}
$$

Taking the mapping cone of $\delta : T^{-1} I_Z^\bullet \to I_X^\bullet$, we get a desired right resolution $\mu_Y : Y \to I_Y^\bullet$ of Y. The last assertion follows by Lemma 2.6.

Proposition 2.8. *Let* $0 \to X \xrightarrow{f} Y \to Z \to 0$ *be an exact sequence in* $\mathscr{A}$ *and* $X \to I_X^\bullet$, $Y \to I_Y^\bullet$ *injective resolutions of X and Y, respectively. Then the following hold.*

(1) *There exists* $\hat{f} \in C(\mathscr{A})(I_X^\bullet, I_Y^\bullet)$ *such that* $H^0(\hat{f}) = f$.

(2) *There exists an injective resolution* $Z \to I_Z^\bullet$ *of Z such that* $C(\hat{f}) \cong C(\mathrm{id}_{I_X^0}) \oplus I_Z^\bullet$ *in* $C(\mathscr{A})$.

Proof. (1) By Lemma 1.8.

(2) Note first that by Proposition 2.4 $H^0(C(\hat{f})) \cong Z$ and $H^n(C(\hat{f})) = 0$ for all $n \neq 0$. We have an exact sequence in $C(\mathscr{A})$ of the form

$$0 \to C(\mathrm{id}_{I_X^0}) \to C(\hat{f}) \to I_Z^\bullet \to 0,$$

which splits because by Proposition 1.6(1) $C(\mathrm{id}_{I_X^0})$ is injective in $C(\mathscr{A})$. Note that $I_Z^n \in \mathscr{I}$ for all $n \in \mathbb{Z}$ and $I_Z^n = 0$ for all $n < 0$. Also, by Proposition 1.3 $H^0(I_Z^\bullet) \cong Z$ and $H^n(I_Z^\bullet) = 0$ for all $n \neq 0$. It follows that $I_Z^\bullet$ is an injective resolution of Z.

Corollary 2.9. *Let* $X \in \mathrm{Ob}(\mathscr{A})$ *and* $n \geq 0$. *Let*

$$0 \to X_n \to \cdots \to X_0 \to X \to 0$$

be an exact sequence in $\mathscr{A}$ *and* $X_i \to I_{X_i}^\bullet$ *an injective resolution of* X_i *for* $0 \leq i \leq n$. *Then there exists a monomorphism* $X \to \bigoplus_{i=0}^n I_{X_i}^i$ *in* $\mathscr{A}$.

Proof. In case $n = 0$, the assertion is obvious. Let $n > 0$ and put $X'_{n-1} = \mathrm{Cok}(X_n \to X_{n-1})$. By Proposition 2.8 there exist $\phi : I_{X_n}^\bullet \to I_{X_{n-1}}^\bullet$ and an injective resolution $X'_{n-1} \to I_{X'_{n-1}}^\bullet$ of X'_{n-1} such that $C(\mathrm{id}_I) \oplus I_{X'_{n-1}}^\bullet \cong C(\phi)$, where $I = I_{X_n}^0$. In case $n = 1$, $X \cong X'_{n-1}$ and $I_{X'_{n-1}}^{n-1}$ is a direct summand of $I_{X_n}^n \oplus I_{X_{n-1}}^{n-1}$, so that X embeds in $I_{X_n}^n \oplus I_{X_{n-1}}^{n-1}$. Let $n > 1$ and assume the assertion is true for $n - 1$. Then $I_{X'_{n-1}}^{n-1} \cong I_{X_n}^n \oplus I_{X_{n-1}}^{n-1}$ and by induction hypothesis X embeds in $(\bigoplus_{i=0}^{n-2} I_{X_i}^i) \oplus I_{X'_{n-1}}^{n-1} \cong \bigoplus_{i=0}^n I_{X_i}^i$.

Proposition 2.10. (1) *Let* $n_0 \in \mathbb{Z}$ *and define an automorphism* $\rho : \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$ *of the identity functor* $\mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ *as follows*: $\rho_X^n = (-1)^{n+n_0}\,\mathrm{id}_{X^n}$ *for all* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *and* $n \in \mathbb{Z}$. *Then* ρ *is an involution of* $\mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$, *i.e.*, $\rho^2 = \mathrm{id}$, *and satisfies* $T\rho = -\rho_T$.

(2) *Let* $\rho : \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$ *be an involution of the identity functor* $\mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$ *such that* $T\rho = -\rho_T$. *Then for any* $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *we have a cochain complex*

$$C_\rho(u) = (TX \oplus Y, \begin{bmatrix} -d_{TX} & 0 \\ Tu \circ \rho_{TX} & d_Y \end{bmatrix})$$

which makes the following diagram in $C(\mathscr{A})$ *commute*

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{\mu} & C_\rho(u) & \xrightarrow{\varepsilon} & T\overline{X}^\bullet \\
\| & & \| & & \| & & \downarrow \rho_{TX} \\
X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{\mu} & C_\rho(u) & \xrightarrow{\varepsilon'} & TX^\bullet \\
\| & & \| & & \downarrow \phi & & \| \\
X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{\mu} & C(u) & \xrightarrow{\varepsilon} & TX^\bullet ,
\end{array}$$

where $\mu = {}^t[0\ \ 1]$, $\varepsilon = [1\ \ 0]$, $\varepsilon' = [\rho_{TX}\ \ 0]$, $\phi = \begin{bmatrix} \rho_{TX} & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\overline{X}^\bullet = (X, -d_X)$.

Proof. Straightforward.

Proposition 2.11. (Dual of Proposition 2.7)**.** *Let* $0 \to X \xrightarrow{\mu} Y \xrightarrow{\varepsilon} Z \to 0$ *be an exact sequence in* $\mathscr{A}$. *Let* $\varepsilon_X : P_X^\bullet \to X$ *be a left resolution of* X *and* $\varepsilon_Z : P_Z^\bullet \to Z$ *a projective resolution* Z. *Then there exists a left resolution* $\varepsilon_Y : P_Y^\bullet \to Y$ *of* Y *such that we have a commutative diagram in* $C(\mathscr{A})$ *with exact rows*

$$\begin{array}{ccccccccc}
0 \to & P_X^\bullet & \xrightarrow{\hat{\mu}} & P_Y^\bullet & \xrightarrow{\hat{\varepsilon}} & P_Z^\bullet & \to 0 \\
& \varepsilon_X \downarrow & & \downarrow \varepsilon_Y & & \downarrow \varepsilon_Z & \\
0 \to & X & \xrightarrow{\mu} & Y & \xrightarrow{\varepsilon} & Z & \to 0 .
\end{array}$$

Furthermore, we may assume $P_Y^n = P_Z^n \oplus P_X^n$ *for all* $n \geq 0$ *and* $\hat{\mu} = {}^t[0\ \ 1]$, $\hat{\varepsilon} = [1\ \ 0]$.

Proposition 2.12 (Dual of Proposition 2.8). *Let* $0 \to X \to Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathscr{A}$ *and* $P_Y^\bullet \to Y$, $P_Z^\bullet \to Z$ *be projective resolutions of* Y *and* Z, *respectively. Then the following hold.*

(1) *There exists* $\hat{g} \in C(\mathscr{A})(P_Y^\bullet, P_Z^\bullet)$ *such that* $H^0(\hat{g}) = g$.

(2) *There exists a projective resolution* $P_X^\bullet \to X$ *of* X *such that* $C(\hat{g}) \cong C(\mathrm{id}_{P_Z^0}) \oplus TP_X^\bullet$ *in* $C(\mathscr{A})$.

Corollary 2.13 (Dual of Corollary 2.9). *Let* $X \in \mathrm{Ob}(\mathscr{A})$ *and* $n \geq 0$. *Let*

$$0 \to X \to X^0 \to \cdots \to X^n \to 0$$

be an exact sequence in $\mathscr{A}$ *and* $P_{X^i}^\bullet \to X^i$ *a projective resolution of* X^i *for each* $0 \leq i \leq n$. *Then there exists an epimorphism* $\bigoplus_{i=0}^n P_{X^i}^{-i} \to X$ *in* $\mathscr{A}$.

§3. Homotopy categories

Throughout this section, $\mathscr{A}$ is an abelian category and $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$. Unless stated otherwise, functors are covariant functors.

Proposition 3.1. *For a morphism* $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *the following are equivalent.*

(1) *There exists* $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet)$ *such that* $u = h \circ d_X + T^{-1}(d_Y \circ h)$.

(2) *The canonical exact sequence* $0 \to Y^\bullet \to C(u) \to TX^\bullet \to 0$ *splits.*

(3) *u factors through* ${}^t[0 \;\; 1] : X^\bullet \to C(\mathrm{id}_X)$.

(4) *u factors through* $[1 \;\; 0] : C(\mathrm{id}_{T^{-1}Y}) \to Y^\bullet$.

(5) *u factors through* $C(\mathrm{id}_Z)$ *for some* $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$.

Proof. The implications (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) are obvious. Also, it follows by Proposition 2.2 that (2) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (2). Let $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet)$ with $u = h \circ d_X + T^{-1}(d_Y \circ h)$. Since

$$[h \quad 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathrm{id}_Y,$$

it suffices to check that $[h \quad 1] : C(u) \to Y^\bullet$ is a morphism in $C(\mathscr{A})$. We have

$$\begin{aligned}[Th \quad 1]\begin{bmatrix} -Td_X & 0 \\ Tf & d_Y \end{bmatrix} &= [Tf - Th \circ Td_X \quad d_Y] \\ &= [d_Y \circ h \quad d_Y] \\ &= d_Y\,[h \quad 1].\end{aligned}$$

(5) $\Rightarrow$ (1). Let $u = h \circ v$ with $v = {}^t[v_1 \;\; v_2] : X^\bullet \to C(\mathrm{id}_Z)$, $h = [h_1 \;\; h_2] : C(\mathrm{id}_Z) \to Y^\bullet$. Then

$$\begin{bmatrix} -Td_Z & 0 \\ 1 & d_Z \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Tv_1 \\ Tv_2 \end{bmatrix} d_X$$

implies $v_1 = Tv_2 \circ d_X - d_Z \circ v_2$, and

$$d_Y\,[h_1 \quad h_2] = [Th_1 \quad Th_2]\begin{bmatrix} -Td_Z & 0 \\ 1 & d_Z \end{bmatrix}$$

implies $Th_2 = d_Y \circ h_1 + Th_1 \circ Td_Z$,. Thus

$$
\begin{aligned}
u &= h_1 \circ v_1 + h_2 \circ v_2 \\
&= h_1 \circ (Tv_2 \circ d_X - d_Z \circ v_2) + (T^{-1}d_Y \circ T^{-1}h_1 + h_1 \circ d_Z) \circ v_2 \\
&= h_1 \circ Tv_2 \circ d_X + T^{-1}d_Y \circ T^{-1}h_1 \circ v_2 \\
&= (h_1 \circ Tv_2) \circ d_X + T^{-1}d_Y \circ T^{-1}(h_1 \circ Tv_2).
\end{aligned}
$$

Definition 3.1. For each pair of $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, we denote by $\mathrm{Htp}(X^\bullet, Y^\bullet)$ the subset of $C(\mathscr{A})(X^\bullet, Y^\bullet)$ consisting of morphisms $u : X^\bullet \to Y^\bullet$ which satisfy the equivalent conditions of Proposition 3.1.

Definition 3.2. Let $u, v \in C(\mathscr{A})(X^\bullet, Y^\bullet)$. Then u is said to be homotopic to v, written $u \simeq v$, if $u - v \in \mathrm{Htp}(X^\bullet, Y^\bullet)$. If $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet)$ satisfies $u - v = h \circ d_X + T^{-1}(d_Y \circ h)$, then h is called a homotopy and written $h : u \simeq v$.

Lemma 3.2. (1) *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, $\mathrm{Htp}(X^\bullet, Y^\bullet)$ *is an additive subgroup of* $C(\mathscr{A})(X^\bullet, Y^\bullet)$.

(2) *For any two consecutive morphisms* $u : X^\bullet \to Y^\bullet$, $v : Y^\bullet \to Z^\bullet$ *in* $C(\mathscr{A})$, *if either* $u \in \mathrm{Htp}(X^\bullet, Y^\bullet)$ *or* $v \in \mathrm{Htp}(Y^\bullet, Z^\bullet)$ *then* $v \circ u \in \mathrm{Htp}(X^\bullet, Z^\bullet)$.

(3) *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, *the translation* T *induces an isomorphism* $\mathrm{Htp}(X^\bullet, Y^\bullet) \tilde{\to} \mathrm{Htp}(TX^\bullet, TY^\bullet)$.

Proof. By Proposition 3.1.

Definition 3.3. According to Lemma 3.2, we can define the residue category $K(\mathscr{A}) = C(\mathscr{A})/\mathrm{Htp}$, called the homotopy category, as follows: $\mathrm{Ob}(K(\mathscr{A})) = \mathrm{Ob}(C(\mathscr{A}))$; and for each pair of objects $X^\bullet, Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ we set

$$K(\mathscr{A})(X^\bullet, Y^\bullet) = C(\mathscr{A})(X^\bullet, Y^\bullet)/\mathrm{Htp}(X^\bullet, Y^\bullet).$$

Then the translation $T : C(\mathscr{A}) \tilde{\to} C(\mathscr{A})$ induces an autofunctor $T : K(\mathscr{A}) \tilde{\to} K(\mathscr{A})$, which is also called the translation. Similarly, for a subcollection $\mathscr{L}$ of $\mathrm{Ob}(\mathscr{A})$ and for * = +, –, b, (+, b), (–, b) or nothing, we define the homotopy category $K^*(\mathscr{L}) = C^*(\mathscr{L})/\mathrm{Htp}$. Then the canonical functor $K^*(\mathscr{L}) \to K(\mathscr{A})$ is fully faithful and $K^*(\mathscr{L})$ can be identified with the full subcategory of $K(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(C^*(\mathscr{L}))$.

Remark 3.1. $\mathrm{Htp}(X, Y) = 0$ for all $X, Y \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$, so that we have a full embedding $\mathscr{A}^{\mathbb{Z}} \to K(\mathscr{A})$.

Lemma 3.3. *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *we have exact sequences*

$$C(\mathscr{A})(C(\mathrm{id}_X),\ Y^\bullet) \to C(\mathscr{A})(X^\bullet,\ Y^\bullet) \to K(\mathscr{A})(X^\bullet,\ Y^\bullet) \to 0,$$

$$C(\mathscr{A})(X^\bullet, C(\mathrm{id}_{T^{-1}Y})) \to C(\mathscr{A})(X^\bullet,\ Y^\bullet) \to K(\mathscr{A})(X^\bullet,\ Y^\bullet) \to 0.$$

Proof. Straightforward.

Proposition 3.4. *Let* $\mathscr{L}$ *be a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *containing zero objects and closed under finite direct sums. Then, for* $* = +, -,$ b *or nothing, the following hold.*

(1) $K^*(\mathscr{L})$ *is an additive category and the canonical functor* $C^*(\mathscr{L}) \to K^*(\mathscr{L})$ *is additive.*

(2) *The canonical functor* $C^*(\mathscr{L}) \to K^*(\mathscr{L})$ *preserves direct products. In particular, if* $\mathscr{A}$ *satisfies the condition Ab3***, then arbitrary direct products exist in* $K(\mathscr{A})$*.*

(3) *The canonical functor* $C^*(\mathscr{L}) \to K^*(\mathscr{L})$ *preserves direct sums. In particular, if* $\mathscr{A}$ *satisfies the condition Ab3, then arbitrary direct sum exist in* $K(\mathscr{A})$*.*

Proof. (1) Immediate by definition.

(2) Let $\{Y^\bullet_\lambda\}_{\lambda \in \Lambda}$ be a family of objects in $C^*(\mathscr{L})$ indexed by a set Λ and assume the direct product $\prod Y^\bullet_\lambda$ exists in $C^*(\mathscr{L})$. Let $X^\bullet \in \mathrm{Ob}(C^*(\mathscr{L}))$. Since by Lemma 3.3 we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
C(\mathscr{A})(C(\mathrm{id}_X), \prod Y^\bullet_\lambda) & \to & C(\mathscr{A})(X^\bullet, \prod Y^\bullet_\lambda) & \to & K(\mathscr{A})(X^\bullet \prod Y^\bullet_\lambda) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod C(\mathscr{A})(C(\mathrm{id}_X),\ Y^\bullet_\lambda) & \to & \prod C(\mathscr{A})(X^\bullet,\ Y^\bullet_\lambda) & \to & \prod K(\mathscr{A})(X^\bullet,\ Y^\bullet_\lambda) & \to & 0,
\end{array}$$

where the vertical maps are canonical ones, it follows that

$$K(\mathscr{A})(X^\bullet, \prod Y^\bullet_\lambda) \xrightarrow{\sim} \prod K(\mathscr{A})(X^\bullet,\ Y^\bullet_\lambda)$$

canonically. The last assertion follows by Proposition 1.11.

(3) Dual of (2).

Proposition 3.5. *For* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) $X^\bullet = 0$ *in* $K(\mathscr{A})$.

(2) $K(\mathscr{A})(X^\bullet,\ X^\bullet) = 0$.

(3) *There exists* $h : \mathrm{id}_X \simeq 0$.

(4) $X^\bullet \cong C(\mathrm{id}_Z)$ *in* $C(\mathscr{A})$ *for* $Z^\bullet = Z^\bullet(X^\bullet)$.

(5) $X^\bullet \cong C(\mathrm{id}_Z)$ *in* $C(\mathscr{A})$ *for some* $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$.

Proof. The implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(3) $\Rightarrow$ (4). Let $Z^\bullet = Z^\bullet(X^\bullet)$. By Proposition 3.1 ${}^t[0 \ \ 1] : X^\bullet \to C(\mathrm{id}_X)$ is a section. Since $C(\mathrm{id}_X)$ is acyclic, so is $X^\bullet$. Thus we may consider that $B^\bullet(X^\bullet) = Z^\bullet$ and $B'^\bullet(X^\bullet) = TZ^\bullet$. Let $j : Z^\bullet \to X^\bullet$ be the inclusion and $p : X^\bullet \to TZ^\bullet$ the epimorphism with $d_X = Tj \circ p$. Note that $p \circ T^{-1}d_X = 0$. Since $\mathrm{id}_X = h \circ d_X + T^{-1}(d_X \circ h)$,

$$\begin{aligned} p &= p \circ (h \circ d_X + T^{-1}(d_X \circ h)) \\ &= p \circ h \circ d_X \\ &= p \circ h \circ Tj \circ p. \end{aligned}$$

Thus, since p is epic, $\mathrm{id}_{TZ} = p \circ h \circ Tj$. Hence, since $p \circ j = 0$, we get an isomorphism in $C(\mathscr{A})$

$$[hT(j) \ \ j] : C(\mathrm{id}_Z) \xrightarrow{\sim} X^\bullet.$$

(5) $\Rightarrow$ (1). By Proposition 3.1.

Proposition 3.6. *For* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) $X^\bullet$ *is injective (resp. projective) in* $C(\mathscr{A})$.

(2) $Z^\bullet = Z^\bullet(X^\bullet)$ *is injective (resp. projective) in* $\mathscr{A}^{\mathbb{Z}}$ *and* $X^\bullet \cong C(\mathrm{id}_Z)$.

Proof. (1) $\Rightarrow$ (2). The canonical exact sequence $0 \to X^\bullet \to C(\mathrm{id}_X) \to TX^\bullet \to 0$ splits, so that by Propositions 3.1 and 3.5 $X^\bullet \cong C(\mathrm{id}_Z)$ with $Z^\bullet = Z^\bullet(X^\bullet)$. In case $X^\bullet$ is injective, so is $SZ^\bullet = C(\mathrm{id}_Z)$. Let $j : Y \to Y'$ be a monomorphism in $\mathscr{A}^{\mathbb{Z}}$ and $f \in \mathscr{A}^{\mathbb{Z}}(Y, Z^\bullet)$. Since Sj is monic, there exists $\hat{g} \in C(\mathscr{A})(SY, SZ^\bullet)$ such that $Sf = \hat{g} \circ Sj$. Note that

$$\hat{g} = \begin{bmatrix} Tg & 0 \\ h & g \end{bmatrix} \text{ with } g \in \mathscr{A}^{\mathbb{Z}}(Y, Z^\bullet),\ h \in \mathscr{A}^{\mathbb{Z}}(TY', Z^\bullet).$$

Thus, since $US = T \oplus \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$, $U(S(f)) = U(\hat{g}) \circ U(S(j))$ implies $f = gj$.

(2) $\Rightarrow$ (1). Since S has an exact left adjoint U, S takes injective objects into injective objects. Thus $C(\mathrm{id}_Z) = SZ^\bullet$ is injective.

Lemma 3.7. *For any* $X \in \mathrm{Ob}(\mathscr{A})$, $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *we have isomorphisms*

$$K(\mathscr{A})(X, Y^\bullet) \cong H^0(\mathscr{A}(X, Y^\bullet)), \quad K(\mathscr{A})(Y^\bullet, X) \cong H^0(\mathscr{A}(Y^\bullet, X)).$$

Proof. We may consider that

$$C(\mathscr{A})(X,\ Y^\bullet) = \{u \in \mathscr{A}(X, Y^0) \mid d_Y^0 \circ u = 0\}$$
$$= Z^0(\mathscr{A}(X,\ Y^\bullet)),$$
$$C(\mathscr{A})(C(\mathrm{id}_X),\ Y^\bullet) = \{(v, u) \in \mathscr{A}(X, Y^{-1}) \times \mathscr{A}(X, Y^0) \mid u = d_Y^{-1} \circ v\}.$$

Then, for $\mu = {}^t[0\ \ 1] : X \to C(\mathrm{id}_X)$, we have $\mathrm{Im}(C(\mathscr{A})(\mu,\ Y^\bullet)) = B^0(\mathscr{A}(X,\ Y^\bullet))$. Thus, since by Lemma 3.3 we have an exact sequence

$$C(\mathscr{A})(C(\mathrm{id}_X),\ Y^\bullet) \to C(\mathscr{A})(X,\ Y^\bullet) \to K(\mathscr{A})(X,\ Y^\bullet) \to 0,$$

we get $K(\mathscr{A})(X,\ Y^\bullet) \cong H^0(\mathscr{A}(X,\ Y^\bullet))$. Dually, we have $K(\mathscr{A})(Y^\bullet, X) \cong H^0(\mathscr{A}(Y^\bullet, X))$.

Proposition 3.8. *For any $X \in \mathrm{Ob}(\mathscr{A})$, $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ and $n \in \mathbb{Z}$, we have isomorphisms*

$$K(\mathscr{A})(X, T^n\ Y^\bullet) \cong H^n(\mathscr{A}(X,\ Y^\bullet)),\quad K(\mathscr{A})(T^{-n}\ Y^\bullet, X) \cong H^n(\mathscr{A}(Y^\bullet, X)).$$

Proof. By Lemma 3.7 and Proposition 1.4 we have

$$\begin{aligned} K(\mathscr{A})(X, T^n\ Y^\bullet) &\cong H^0(\mathscr{A}(X, T^n\ Y^\bullet)) \\ &\cong H^0(T^n\,\mathscr{A}(X,\ Y^\bullet)) \\ &\cong H^n(\mathscr{A}(X,\ Y^\bullet)). \end{aligned}$$

Dually, we have $K(\mathscr{A})(T^{-n}\ Y^\bullet, X) \cong H^n(\mathscr{A})(Y^\bullet, X))$.

Proposition 3.9. *The functor $H^\bullet : C(\mathscr{A}) \to \mathscr{A}^{\mathbb{Z}}$ factors through $K(\mathscr{A})$.*

Proof. Let $u \in \mathrm{Htp}(X^\bullet,\ Y^\bullet)$. Then u factors through some $C(\mathrm{id}_Z)$. Since by Proposition 2.3 $C(\mathrm{id}_Z)$ is acyclic, it follows that $H^\bullet(u) = 0$.

Remark 3.2. The converse of Proposition 3.9 fails, i.e., for $u \in C(\mathscr{A})(X^\bullet,\ Y^\bullet)$, $H^\bullet(u) = 0$ does not necessarily imply $u \simeq 0$. Let $f \in \mathscr{A}(X, Y)$ and $\mu = {}^t[0\ \ 1] : Y \to C(f)$. Then $H^0(\mu)$ is just the canonical epimorphism $Y \to \mathrm{Cok}\, f$. Also, $\mu \simeq 0$ if and only if f is a retraction. Thus, in case f is an epimorphism and not a retraction, we have $H^\bullet(u) = 0$ and $\mu \not\simeq 0$.

Proposition 3.10. *Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. Then the following hold.*

(1) *F is extended to an additive functor $F : K(\mathscr{A}) \to K(\mathscr{B})$ which commutes with the translation.*

(2) *Assume F has a right (resp. left) adjoint $G : \mathscr{B} \to \mathscr{A}$. Then the extended functor G :*

$K(\mathscr{B}) \to K(\mathscr{A})$ *is a right (resp. left) adjoint of* $F : K(\mathscr{A}) \to K(\mathscr{B})$. *Furthermore, if* $G : \mathscr{B} \to \mathscr{A}$ *is fully faithful, so is* $G : K(\mathscr{B}) \to K(\mathscr{A})$.

Proof. (1) It is obvious that the extended functor $F : C(\mathscr{A}) \to C(\mathscr{B})$ commutes with the translation. Let $X^\bullet$, $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. Since $F(C(\mathrm{id}_Z)) \cong C(\mathrm{id}_{FZ})$ for all $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, $Fu \in \mathrm{Htp}(FX^\bullet, FY^\bullet)$ for all $u \in \mathrm{Htp}(X^\bullet, Y^\bullet)$.

(2) Let $G : \mathscr{B} \to \mathscr{A}$ be a right adjoint of $F : \mathscr{A} \to \mathscr{B}$ and let $\varepsilon: \mathbf{1}_{\mathscr{A}} \to GF$, $\delta : FG \to \mathbf{1}_{\mathscr{B}}$ be the unit and the counit, respectively. Then ε, δ are extended to homomorphisms of functors $\varepsilon: \mathbf{1}_{K(\mathscr{A})} \to GF$, $\delta : FG \to \mathbf{1}_{K(\mathscr{B})}$, respectively. It is easy to see that the equations $\delta_F \circ F\varepsilon = \mathrm{id}_F$, $G\delta \circ \varepsilon_G = \mathrm{id}_G$ are also satisfied by extended functors. Furthermore, if $\delta : FG \to \mathbf{1}_{\mathscr{B}}$ is an isomorphism, so is $\delta : FG \to \mathbf{1}_{K(\mathscr{B})}$.

Proposition 3.11. *For any* $u, v \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *the following hold.*

(1) *For* $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet)$, $h : u \simeq v$ *if and only if*

$$\phi = \begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix} \in C(\mathscr{A})(C(u), C(v)).$$

(2) *There exists* $h : u \simeq v$ *if and only if there exists* $\phi \in C(\mathscr{A})(C(u), C(v))$ *which makes the following diagram commute*

$$\begin{array}{ccccccccc} 0 & \to & Y^\bullet & \xrightarrow{\mu} & C(u) & \xrightarrow{\varepsilon} & TX^\bullet & \to & 0 \\ & & \| & & \downarrow \phi & & \| & & \\ 0 & \to & Y^\bullet & \xrightarrow{\mu} & C(v) & \xrightarrow{\varepsilon} & TX^\bullet & \to & 0, \end{array}$$

where $\mu = {}^{t}[0 \ \ 1]$ *and* $\varepsilon = [1 \ \ 0]$.

Proof. (1) Straightforward.

(2) Note that $\phi \in \mathscr{A}^{\mathbb{Z}}(C(u), C(v))$ makes the diagram commute if and only if it is of the form

$$\phi = \begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix} \text{ with } h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet).$$

The assertion follows by the part (1).

Lemma 3.12. *Let* $\mu_X : X \to I_X^\bullet$ *be a right resolution of* $X \in \mathrm{Ob}(\mathscr{A})$ *and* $\mu_Y : Y \to I_Y^\bullet$ *an injective resolution of* $Y \in \mathrm{Ob}(\mathscr{A})$. *Then* $H^0 : K(\mathscr{A}) \to \mathscr{A}$ *induces an isomorphism*

$$K(\mathscr{A})(I_X^\bullet,\ I_Y^\bullet) \xrightarrow{\sim} \mathscr{A}(X, Y),\ \phi \mapsto H^0(\phi).$$

Proof. By Lemma 1.8.

Proposition 3.13. *Assume $\mathscr{A}$ has enough injectives. Choose arbitrarily an injective resolution $\mu_X : X \to I_X^\bullet$ of each $X \in$ Ob($\mathscr{A}$). Then $I_X^\bullet \in$ Ob($K(\mathscr{A})$) is uniquely determined up to isomorphisms and we get a full embedding*

$$\mathscr{A} \to K(\mathscr{A}),\ X \mapsto I_X^\bullet.$$

Proof. Let $X \to I^\bullet$ be another injective resolution of $X \in$ Ob($\mathscr{A}$). By Lemma 1.8 there exist $\phi \in C(\mathscr{A})(I_X^\bullet,\ I^\bullet)$, $\psi \in C(\mathscr{A})(I^\bullet,\ I_X^\bullet)$ such that $H^0(\phi) = H^0(\psi) = \mathrm{id}_X$. Then $H^0(\psi \circ \phi) = H^0(\phi \circ \psi) = \mathrm{id}_X$ and it follows by Lemma 3.12 that $\psi \circ \phi = \phi \circ \psi = \mathrm{id}_X$ in $K(\mathscr{A})$. Thus ϕ is an isomorphism in $K(\mathscr{A})$. The last assertion follows by Lemma 3.12.

Proposition 3.14. *Let*

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{\mu} & Y & \xrightarrow{\varepsilon} & Z & \to & 0 \\ & & f \downarrow & & \downarrow g & & \downarrow h & & \\ 0 & \to & X' & \xrightarrow{\mu'} & Y' & \xrightarrow{\varepsilon'} & Z' & \to & 0 \end{array}$$

be a commutative diagram in $\mathscr{A}$ with exact rows and let

$$0 \to I_X^\bullet \xrightarrow{\hat{\mu}} I_Y^\bullet \xrightarrow{\hat{\varepsilon}} I_Z^\bullet \to 0, \qquad 0 \to I_{X'}^\bullet \xrightarrow{\hat{\mu}'} I_{Y'}^\bullet \xrightarrow{\hat{\varepsilon}'} I_{Z'}^\bullet \to 0$$

be exact sequences of injective resolutions over the top and the bottom rows, respectively. Then for any $\hat{f} : I_X^\bullet \to I_{X'}^\bullet$ over f and $\hat{h} : I_Z^\bullet \to I_{Z'}^\bullet$ over h, the following hold.

(1) *There exists $\hat{g} : I_Y^\bullet \to I_{Y'}^\bullet$ over g which makes the following diagram commute*

$$\begin{array}{ccccccccc} 0 & \to & I_X^\bullet & \xrightarrow{\hat{\mu}} & I_Y^\bullet & \xrightarrow{\hat{\varepsilon}} & I_Z^\bullet & \to & 0 \\ & & \hat{f} \downarrow & & \downarrow \hat{g} & & \downarrow \hat{h} & & \\ 0 & \to & I_{X'}^\bullet & \xrightarrow{\hat{\mu}'} & I_{Y'}^\bullet & \xrightarrow{\hat{\varepsilon}'} & I_{Z'}^\bullet & \to & 0\,. \end{array}$$

(2) *In case $f = g = h = 0$, for any $u : \hat{f} \simeq 0$ and $w : \hat{h} \simeq 0$ there exists $v : \hat{g} \simeq 0$ which makes the following diagram in $\mathscr{A}^{\mathbb{Z}}$ commute*

$$\begin{array}{ccccccccc}
0 \to & T(I_X^\bullet) & \xrightarrow{T\hat{\mu}} & T(I_Y^\bullet) & \xrightarrow{T\hat{\varepsilon}} & T(I_Z^\bullet) & \to & 0 \\
 & u \downarrow & & \downarrow v & & \downarrow w & & \\
0 \to & I_{X'}^\bullet & \xrightarrow{\hat{\mu}'} & I_{Y'}^\bullet & \xrightarrow{\hat{\varepsilon}'} & I_{Z'}^\bullet & \to & 0 .
\end{array}$$

Proof. (1) According to Proposition 2.7, we may assume

$$I_Y^\bullet = C(\delta) \text{ with } \delta \in C(\mathscr{A})(T^{-1} I_Z^\bullet,\ I_X^\bullet), \quad I_{Y'}^\bullet = C(\delta') \text{ with } \delta' \in C(\mathscr{A})(T^{-1} I_{Z'}^\bullet,\ I_{X'}^\bullet),$$

$$\mu_Y = \begin{bmatrix} \mu_Z \circ \varepsilon \\ \delta^{-1} \end{bmatrix} \text{ with } \delta^{-1} \in \mathscr{A}(Y,\ I_X^0), \quad \mu_{Y'} = \begin{bmatrix} \mu_{Z'} \circ \varepsilon' \\ \delta'^{-1} \end{bmatrix} \text{ with } \delta'^{-1} \in \mathscr{A}(Y',\ I_{X'}^0),$$

$$\hat{\mu} = {}^t[0\ \ 1], \quad \hat{\varepsilon} = [1\ \ 0], \quad \hat{\mu}' = {}^t[0\ \ 1] \quad \text{and} \quad \hat{\varepsilon}' = [1\ \ 0].$$

Then we have the following commutative diagrams with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu} & Y & \xrightarrow{-\mu_Z \circ \varepsilon} & I_Z^0 & \xrightarrow{-d_{I_Z}^0} & I_Z^1 & \xrightarrow{-d_{I_Z}^1} & \cdots \\
 & & \| & & \downarrow \delta^{-1} & & \downarrow \delta^0 & & \downarrow \delta^1 & & \\
0 & \longrightarrow & X & \xrightarrow{\mu_X} & I_X^0 & \xrightarrow{d_{I_X}^0} & I_X^1 & \xrightarrow{d_{I_X}^1} & I_X^2 & \xrightarrow{d_{I_X}^2} & \cdots \\
 & & f \downarrow & & \downarrow \hat{f}^0 & & \downarrow \hat{f}^1 & & \downarrow \hat{f}^2 & & \\
0 & \longrightarrow & X' & \xrightarrow{\mu_{X'}} & I_{X'}^0 & \xrightarrow{d_{I_{X'}}^0} & I_{X'}^1 & \xrightarrow{d_{I_{X'}}^1} & I_{X'}^2 & \xrightarrow{d_{I_{X'}}^2} & \cdots ,
\end{array}$$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X & \xrightarrow{\mu} & Y & \xrightarrow{-\mu_Z \circ \varepsilon} & I_Z^0 & \xrightarrow{-d_{I_Z}^0} & I_Z^1 & \xrightarrow{-d_{I_Z}^1} & \cdots \\
 & & f \downarrow & & \downarrow g & & \downarrow \hat{h}^0 & & \downarrow \hat{h}^1 & & \\
0 & \longrightarrow & X' & \xrightarrow{\mu'} & Y' & \xrightarrow{-\mu_{Z'} \circ \varepsilon'} & I_{Z'}^0 & \xrightarrow{-d_{I_{Z'}}^0} & I_{Z'}^1 & \xrightarrow{-d_{I_{Z'}}^1} & \cdots \\
 & & \| & & \downarrow \delta'^{-1} & & \downarrow \delta'^0 & & \downarrow \delta'^1 & & \\
0 & \longrightarrow & X' & \xrightarrow{\mu_{X'}} & I_{X'}^0 & \xrightarrow{d_{I_{X'}}^0} & I_{X'}^1 & \xrightarrow{d_{I_{X'}}^1} & I_{X'}^2 & \xrightarrow{d_{I_{X'}}^2} & \cdots .
\end{array}$$

Thus, setting

$$\psi^{-1} = \hat{f}^0 \circ \delta^{-1} - \delta'^{-1} \circ g : Y \to I_{X'}^0,$$
$$\psi^n = \hat{f}^{n+1} \circ \delta^n - \delta'^n \circ \hat{h}^n : I_Z^n \to I_{X'}^{n+1} \text{ for } n \geq 0,$$

we get a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X & \xrightarrow{\mu} & Y & \xrightarrow{-\mu_Z \circ \varepsilon} & I_Z^0 & \xrightarrow{-d_{I_Z}^0} & I_Z^1 & \xrightarrow{-d_{I_Z}^1} & \cdots \\ & & {\scriptstyle 0}\downarrow & & \downarrow \psi^{-1} & & \downarrow \psi^0 & & \downarrow \psi^1 & & \\ 0 & \longrightarrow & X' & \xrightarrow{\mu_{X'}} & I_{X'}^0 & \xrightarrow{d_{I_{X'}}^0} & I_{X'}^1 & \xrightarrow{d_{I_{X'}}^1} & I_{X'}^2 & \xrightarrow{d_{I_{X'}}^2} & \cdots \ . \end{array}$$

It follows by Lemma 1.8 that there exist $\varphi^n : I_Z^n \to I_{X'}^n$ for $n \geq 0$ such that $\psi^{-1} = -\varphi^0 \circ \mu_Z \circ \varepsilon$ and $\psi^n = d_{I_{X'}}^n \circ \varphi^n - \varphi^{n+1} \circ d_{I_Z}^n$ for all $n \geq 0$. Hence, setting

$$\hat{g}^n = \begin{bmatrix} \hat{h}^n & 0 \\ \varphi^n & \hat{f}^n \end{bmatrix} : I_Z^n \oplus I_X^n \to I_{Z'}^n \oplus I_{X'}^n$$

for $n \geq 0$, we get a desired morphism $\hat{g} : I_Y^\bullet \to I_{Y'}^\bullet$.

(2) Put $\varphi' = \varphi - (u \circ T\delta + \delta' \circ T^{-1}w)$. It is not difficult to check that $\varphi' \in C(\mathscr{A})(I_Z^\bullet, I_{X'}^\bullet)$ and $\varphi'^0 \circ \mu_Z = 0$. Thus by Lemma 1.8 there exists $v' : \varphi' \simeq 0$ and we get a desired homotopy

$$v = \begin{bmatrix} w & 0 \\ v' & u \end{bmatrix} : \hat{g} \simeq 0.$$

Lemma 3.15 (Dual of Lemma 3.12). *Let $\varepsilon_X : P_X^\bullet \to X$ be a projective resolution of $X \in$* Ob($\mathscr{A}$) *and $\varepsilon_Y : P_Y^\bullet \to Y$ a left resolution of $Y \in$* Ob($\mathscr{A}$). *Then H^0: $K(\mathscr{A}) \to \mathscr{A}$ induces an isomorphism*

$$K(\mathscr{A})(P_X^\bullet, P_Y^\bullet) \xrightarrow{\sim} \mathscr{A}(X, Y), \phi \mapsto H^0(\phi).$$

Proposition 3.16 (Dual of Proposition 3.13). *Assume $\mathscr{A}$ has enough projectives. Choose arbitrarily a projective resolution $\varepsilon_X : P_X^\bullet \to X$ of each $X \in$* Ob($\mathscr{A}$). *Then $P_X^\bullet \in$* Ob($K(\mathscr{A})$) *is uniquely determined up to isomorphisms and we get a full embedding*

$$\mathscr{A} \to K(\mathscr{A}), X \mapsto P_X^\bullet.$$

Proposition 3.17 (Dual of Proposition 3.14). *Let*

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{\mu} & Y & \xrightarrow{\varepsilon} & Z & \to & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle g} & & \downarrow {\scriptstyle h} & & \\ 0 & \to & X' & \xrightarrow{\mu'} & Y' & \xrightarrow{\varepsilon'} & Z' & \to & 0 \end{array}$$

be a commuta diagram in $\mathscr{A}$ with exact rows and let

$$0 \to P_X^\bullet \xrightarrow{\mu} P_Y^\bullet \xrightarrow{\varepsilon} P_Z^\bullet \to 0, \qquad 0 \to P_{X'}^\bullet \xrightarrow{\hat{\mu}'} P_{Y'}^\bullet \xrightarrow{\hat{\varepsilon}'} P_{Z'}^\bullet \to 0$$

be exact sequences of projective resolutions over the top and the bottom rows, respectively. Then for any $\hat{f} : P_X^\bullet \to P_{X'}^\bullet$ *over* f *and* $\hat{h} : P_Z^\bullet \to P_{Z'}^\bullet$ *over* h, *the following hold.*

(1) *There exists* $\hat{g} : P_Y^\bullet \to P_{Y'}^\bullet$ *over* g *which makes the following diagram commute*

$$\begin{array}{ccccccccc} 0 & \to & P_X^\bullet & \xrightarrow{\hat{\mu}} & P_Y^\bullet & \xrightarrow{\hat{\varepsilon}} & P_Z^\bullet & \to & 0 \\ & & \hat{f} \downarrow & & \downarrow \hat{g} & & \downarrow \hat{h} & & \\ 0 & \to & P_{X'}^\bullet & \xrightarrow{\hat{\mu}'} & P_{Y'}^\bullet & \xrightarrow{\hat{\varepsilon}'} & P_{Z'}^\bullet & \to & 0 . \end{array}$$

(2) *In case* $f = g = h = 0$, *for any* $u : \hat{f} \simeq 0$ *and* $w : \hat{h} \simeq 0$ *there exists* $v : \hat{g} \simeq 0$ *which makes the following diagram in* $\mathscr{A}^{\mathbb{Z}}$ *commute*

$$\begin{array}{ccccccccc} 0 & \to & T(I_X^\bullet) & \xrightarrow{T\hat{\mu}} & T(I_Y^\bullet) & \xrightarrow{T\hat{\varepsilon}} & T(I_Z^\bullet) & \to & 0 \\ & & u \downarrow & & \downarrow v & & \downarrow w & & \\ 0 & \to & I_{X'}^\bullet & \xrightarrow{\hat{\mu}'} & I_{Y'}^\bullet & \xrightarrow{\hat{\varepsilon}'} & I_{Z'}^\bullet & \to & 0 . \end{array}$$

§4. Quasi-isomorphisms

Throughout this section, $\mathscr{A}$ is an abelian category and $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$.

Proposition 4.1. *The following conditions for* $u \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ *do not depend on the choice of a representative of* u *in* $C(\mathscr{A})(X^\bullet, Y^\bullet)$ *and are equivalent.*

(1) $H^\bullet(u)$ *is an isomorphism.*

(2) $C(u)$ *is acyclic.*

Proof. According to Propositions 3.9 and 3.11, both the conditions do not depend on the choice of a representative of u in $C(\mathscr{A})(X^\bullet, Y^\bullet)$. It follows by Propoaition 2.4 that (1) and (2) are equivalent for $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$.

Definition 4.1. A morphism $u \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ is called a quasi-isomorphism if it satisfies the equivalent conditions of Proposition 4.1. We also call a morphism u in $C(\mathscr{A})$ a quasi-isomorphism if it represents a quasi-isomorphism in $K(\mathscr{A})$.

Proposition 4.2. *For any two consecutive morphisms* $u: X^\bullet \to Y^\bullet$, $v: Y^\bullet \to Z^\bullet$ *in* $K(\mathscr{A})$ *the following hold.*

(1) *If two of* u, vu *and* v *are quasi-isomorphisms, then the rest is also a quasi-isomorphism.*

(2) *If two of* $C(u)$, $C(vu)$ *and* $C(v)$ *are acyclic, then the rest is also acyclic.*

Proof. (1) Since $H^\bullet(v) \circ H^\bullet(u) = H^\bullet(vu)$, if two of $H^\bullet(u)$, $H^\bullet(vu)$ and $H^\bullet(v)$ are isomorphisms, then the rest is also an isomorphism.

(2) By the part (1) and Proposition 4.1.

Proposition 4.3. *For any exact sequence* $0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$ *in* $C(\mathscr{A})$ *the following hold.*

(1) v *is a quasi-isomorphism if and only if* $X^\bullet$ *is acyclic.*

(2) u *is a quasi-isomorphism if and only if* $Z^\bullet$ *is acyclic.*

(3) $[0 \quad v] : C(u) \to Z^\bullet$ *is a quasi-isomorphism.*

(4) ${}^t[Tu \quad 0] : TX^\bullet \to C(v)$ *is a quasi-isomorphism.*

Proof. (1) and (2) By Proposition 1.3.

(3) By Proposition 2.5 we have an exact sequence

$$0 \to C(\mathrm{id}_X) \xrightarrow{\phi} C(u) \xrightarrow{\pi} Z^\bullet \to 0,$$

where $\phi = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$, $\pi = [0 \quad v]$. Then by Propositions 2.3 and 2.4 π is a quasi-isomorphism.

(4) Dual of (3).

Lemma 4.4. *Let* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. *Then* $K(\mathscr{A})(-, I^\bullet)$ *vanishes on the acyclic complexes. In particular, if* $I^\bullet$ *is acyclic, then* $I^\bullet = 0$ *in* $K(\mathscr{A})$.

Proof. Let $u \in K(\mathscr{A})(X^\bullet, I^\bullet)$ with $X^\bullet$ acyclic. We will construct a homotopy $h : u \simeq 0$. We may assume $I^n = 0$ for all $n < 0$. Let $h^{-1} = 0 : X^0 \to I^{-1}$. Then $(u^0 - d_I^{-1} \circ h^{-1}) \circ d_X^{-1} = 0$. Thus, the following Claim enables us to make use of induction to construct a desired homotopy $h : u \simeq 0$.

Claim: Let $n \geq 0$ and assume that $h^{n-1} : X^n \to I^{n-1}$ satisfies $(u^n - d_I^{n-1} \circ h^{n-1}) \circ d_X^{n-1} = 0$. Then there exists $h^n : X^{n+1} \to I^n$ such that

$$u^n = h^n \circ d_X^n + d_I^{n-1} \circ h^{n-1} \quad \text{and} \quad (u^{n+1} - d_I^n \circ h^n) \circ d_X^n = 0.$$

Proof. Since $u^n - d_I^{n-1} \circ h^{n-1}$ factors through $Z'^n(X^\bullet) = B^n(X^\bullet)$, and since I^n is injective, $u^n - d_I^{n-1} \circ h^{n-1}$ factors through d_X^n. Thus there exists $h^n : X^{n+1} \to I^n$ with $u^n - d_I^{n-1} \circ h^{n-1} = h^n \circ d_X^n$. Then we have

$$\begin{aligned}(u^{n+1} - d_I^n \circ h^n) \circ d_X^n &= u^{n+1} \circ d_X^n - d_I^n \circ (u^n - d_I^{n-1} \circ h^{n-1}) \\ &= u^{n+1} \circ d_X^n - d_I^n \circ u^n \\ &= 0.\end{aligned}$$

Proposition 4.5. *Let* $s \in K(\mathscr{A})(I^\bullet, X^\bullet)$ *be a quasi-isomorphism with* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. *Then s has a left inverse* $t \in K(\mathscr{A})(X^\bullet, I^\bullet)$ *which is also a quasi-isomorphism*.

Proof. Let $s \in C(\mathscr{A})(I^\bullet, X^\bullet)$ be a quasi-isomorphism with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. We claim that $\mathrm{id}_I - ts \in \mathrm{Htp}(I^\bullet, I^\bullet)$ for some $t \in C(\mathscr{A})(X^\bullet, I^\bullet)$. By Proposition 2.2 we have a push-out diagram in $C(\mathscr{A})$

$$\begin{array}{ccccccccc} 0 & \to & I^\bullet & \to & C(\mathrm{id}_I) & \xrightarrow{v} & TI^\bullet & \to & 0 \\ & & {\scriptstyle s}\downarrow & \mathrm{PO} & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \to & X^\bullet & \xrightarrow{u} & C(s) & \xrightarrow{w} & TI^\bullet & \to & 0. \end{array}$$

Since $C(s)$ is acyclic, by Lemma 4.4 $w \in \mathrm{Htp}(C(s), TI^\bullet)$. Thus by Proposition 3.1 w factors through v and we get a push-out diagram

$$\begin{array}{ccccccccc}
0 & \to & X^\bullet & \xrightarrow{u} & C(s) & \xrightarrow{w} & TI^\bullet & \to & 0 \\
 & & {\scriptstyle t}\downarrow & \text{PO} & \downarrow{\scriptstyle g} & & \| & & \\
0 & \to & I^\bullet & \to & C(\mathrm{id}_I) & \xrightarrow{v} & TI^\bullet & \to & 0.
\end{array}$$

Composing these diagrams, we get a push-out diagram

$$\begin{array}{ccccccccc}
0 & \to & I^\bullet & \to & C(\mathrm{id}_I) & \xrightarrow{v} & TI^\bullet & \to & 0 \\
 & & {\scriptstyle ts}\downarrow & \text{PO} & \downarrow{\scriptstyle gf} & & \| & & \\
0 & \to & I^\bullet & \to & C(\mathrm{id}_I) & \xrightarrow{v} & TI^\bullet & \to & 0.
\end{array}$$

Thus by Propositions 2.2 and 3.11 $\mathrm{id}_I - ts \in \mathrm{Htp}(I^\bullet, I^\bullet)$, i.e., t is a left inverse of s. It follows by Proposition 4.2 that t is a quasi-isomorphism.

Corollary 4.6. *Let* $s \in K(\mathscr{A})(I^\bullet, I'^\bullet)$ *be a quasi-isomorphism with* $I^\bullet, I'^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. *Then* s *is an isomorphism*.

Proof. By Proposition 4.5 s has a left inverse $t \in K(\mathscr{A})(I'^\bullet, I^\bullet)$. Then again by Proposition 4.5 t has a left inverse $s' \in K(\mathscr{A})(I^\bullet, I'^\bullet)$. Thus $s = s'$ and s is an isomorphism.

Proposition 4.7. *Let* $\mathscr{L}$ *be a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *such that for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $X \to I$ *in* $\mathscr{A}$ *with* $I \in \mathscr{L}$. *Then for any* $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$ *there exists a monomorphism* $X^\bullet \to I^\bullet$ *in* $C(\mathscr{A})$ *with* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{L}))$ *which is a quasi-isomrphism.*

Proof. We may assume $X^n = 0$ for all $n < 0$. For each n 0, we have an exact sequence of the form

$$0 \to H^n(X^\bullet) \to Z'^n(X^\bullet) \xrightarrow{d'^n} X^{n+1} \to Z'^{n+1}(X^\bullet) \to 0.$$

We have a monomorphism $u^0 : X^0 \to I^0$ with $I^0 \in \mathscr{L}$. Since $Z'^0(X^\bullet) = X^0$, by putting $Z'^0 = I^0$, we get a monomorphism $u^0 : Z'^0(X^\bullet) \to Z'^0$. Thus the following provides a desired morphism $u : X^\bullet \to I^\bullet$.

Claim: Let $n \geq 0$ and $u^n : Z'^n(X^\bullet) \to Z'^n$ a monomorphism. Then there exists a commutative diagram with exact rows

$$0 \to H^n(X^\bullet) \to Z'^n(X^\bullet) \xrightarrow{d'^n} X^{n+1} \to Z'^{n+1}(X^\bullet) \to 0$$

$$
\begin{array}{ccccccccc}
 & & \| & & u^n\downarrow & \text{PO} & \downarrow v^{n+1} & & \| \\
0 & \to & H^n(X^\bullet) & \to & Z'^n & \to & Y^{n+1} & \to & Z'^{n+1}(X^\bullet) \to 0 \\
 & & \| & & \| & & \downarrow w^{n+1} & & \downarrow u^{n+1} \\
0 & \to & H^n(X^\bullet) & \to & Z'^n & \xrightarrow{f^n} & I^{n+1} & \xrightarrow{g^{n+1}} & Z'^{n+1} \to 0,
\end{array}
$$

where v^{n+1}, w^{n+1} and u^{n+1} are monomorphisms and $I^{n+1} \in \mathscr{L}$.

Proof. Straightforward.

Lemma 4.8 (Dual of Lemma 4.4). *Let* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$. *Then* $K(\mathscr{A})(P^\bullet, -)$ *vanishes on the acyclic complexes*. *In particular, if* $P^\bullet$ *is acyclic, then* $P^\bullet = 0$ *in* $K(\mathscr{A})$.

Proposition 4.9 (Dual of Proposition 4.5). *Let* $s \in K(\mathscr{A})(X^\bullet, P^\bullet)$ *be a quasi-isomorphism with* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$. *Then s has a right inverse* $t \in K(\mathscr{A})(P^\bullet, X^\bullet)$ *which is a quasi-isomorphism*.

Corollary 4.10 (Dual of Corollary 4.6). *Let* $s \in K(\mathscr{A})(P'^\bullet, P^\bullet)$ *be a quasi-isomorphism with* $P^\bullet, P'^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$. *Then s is an isomorphism*.

Proposition 4.11 (Dual of Proposition 4.7). *Let* $\mathscr{L}$ *be a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *such that for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists an epimorphism* $P \to X$ *in* $\mathscr{A}$ *with* $P \in \mathscr{L}$. *Then for any* $X^\bullet \in \mathrm{Ob}(K^-(\mathscr{A}))$ *there exists an epimorphism* $P^\bullet \to X^\bullet$ *in* $C(\mathscr{A})$ *with* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{L}))$ *which is a quasi-isomrphism*.

§5. Mapping cylinders

Throughout this section, $\mathscr{A}$ is an abelian category. Unless otherwise stated, functors are covariant functors.

Definition 5.1. Let $\mathscr{C}$ be a category with an autofunctor $T : \mathscr{C} \tilde{\to} \mathscr{C}$. A cylinder in $\mathscr{C}$ is a sextuple (X, Y, Z, u, v, w) of $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ and $u \in \mathscr{C}(X, Y)$, $v \in \mathscr{C}(Y, Z)$, $w \in \mathscr{C}(Z, TX)$. A homomorphism of cylinders

$$(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$$

is a triple (f, g, h) of $f \in \mathscr{C}(X, X')$, $g \in \mathscr{C}(Y, Y')$, $h \in \mathscr{C}(Z, Z')$ which make the following diagram commute

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'. \end{array}$$

Definition 5.2. For any $u \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ we have a cylinder $(X^\bullet, Y^\bullet, C(u), u, v, w)$ in $K(\mathscr{A})$, where $v = {}^t[0\ \ 1] : Y^\bullet \to C(u)$ and $w = [1\ \ 0] : C(u) \to TX^\bullet$, which we call the mapping cylinder of u.

Definition 5.3. A cylinder $(X^\bullet, Y^\bullet, Z^\bullet, u, v, w)$ in $K(\mathscr{A})$ is called a (distinguished) triangle if it is isomorphic to some mapping cylinder.

Proposition 5.1. *The mapping cylinder of $u \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ does not depend on the choice of a representative of u in $C(\mathscr{A})(X^\bullet, Y^\bullet)$.*

Proof. Let $u, u' \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ with $u = u'$ in $K(\mathscr{A})$, i.e., $u - u' \in \mathrm{Htp}(X^\bullet, Y^\bullet)$. Then by Proposition 3.11 we have an isomorphism of mapping cylinders

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{u} & Y^\bullet & \to & C(u) & \to & TX^\bullet \\ \| & & \| & & \downarrow & & \| \\ X^\bullet & \xrightarrow{u'} & Y^\bullet & \to & C(u') & \to & TX^\bullet. \end{array}$$

Proposition 5.2. *For any triangle $(X^\bullet, Y^\bullet, Z^\bullet, u, v, w)$ we have a long exact sequence*

$$\cdots \to H^n(X^\bullet) \to H^n(Y^\bullet) \to H^n(Z^\bullet) \to H^{n+1}(X^\bullet) \to \cdots .$$

Proof. By Propositions 2.4 and 3.9.

Proposition 5.3. *For any* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, $(X^\bullet, X^\bullet, 0, \mathrm{id}_X, 0, 0)$ *is a triangle*.

Proof. By definition, $C(\mathrm{id}_X) = 0$ in $K(\mathscr{A})$ for all $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$.

Proposition 5.4. *For any mapping cylinder* $(X^\bullet, Y^\bullet, C(u), u, v, w)$ *the following hold*.
(1) $(Y^\bullet, C(u), TX^\bullet, v, w, -Tu)$ *is isomorphic to the mapping cylinder of* v.
(2) $(T^{-1}(C(u)), X^\bullet, Y^\bullet, -T^{-1}w, u, v)$ *is isomorphic to the mapping cylinder of* $-T^{-1}w$.

Proof. (1) Put

$$\hat{h} = \begin{bmatrix} -Tu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} : TX^\bullet \oplus C(\mathrm{id}_Y) \to C(v),$$

$$\hat{g} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & Tu & 0 \\ 0 & 0 & 1 \end{bmatrix} : C(v) \to TX^\bullet \oplus C(\mathrm{id}_Y).$$

Then $\hat{h}$, $\hat{g}$ are isomorphisms in $C(\mathscr{A})$ with $\hat{g} = \hat{h}^{-1}$. Put

$$\hat{w} = \begin{bmatrix} 1 & 0 \\ Tu & 0 \\ 0 & 1 \end{bmatrix} : C(u) \to TX^\bullet \oplus C(\mathrm{id}_Y),$$

$$\hat{u} = \begin{bmatrix} -Tu & 1 & 0 \end{bmatrix} : TX^\bullet \oplus C(\mathrm{id}_Y) \to TY^\bullet.$$

Then the following diagram in $C(\mathscr{A})$ commutes

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{v} & C(u) & \xrightarrow{\hat{w}} & TX^\bullet \oplus C(\mathrm{id}_Y) & \xrightarrow{\hat{u}} & TY^\bullet \\ \| & & \| & & \downarrow \hat{h} & & \| \\ Y^\bullet & \xrightarrow{v} & C(u) & \xrightarrow{w'} & C(v) & \xrightarrow{u'} & TY^\bullet , \end{array}$$

where $w' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} : C(u) \to C(v)$, $u' = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} : C(v) \to TY^\bullet$. Thus we get the following commutative diagram in $K(\mathscr{A})$

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{v} & C(u) & \xrightarrow{w} & TX^\bullet & \xrightarrow{-Tu} & TY^\bullet \\ \| & & \| & & \downarrow h & & \| \\ Y^\bullet & \xrightarrow{v} & C(u) & \xrightarrow{w'} & C(v) & \xrightarrow{u'} & TY^\bullet , \end{array}$$

where $h = {}^t[-Tu \quad 1 \quad 0]$ is an isomorphism in $K(\mathscr{A})$.

(2) Put

$$\hat{h} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & u \\ 0 & 0 & -1 \end{bmatrix} : Y^\bullet \oplus C(\mathrm{id}_X) \to C(-T^{-1}w),$$

$$\hat{g} = \begin{bmatrix} 0 & 1 & u \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} : C(-T^{-1}w) \to Y^\bullet \oplus C(\mathrm{id}_X).$$

Then $\hat{h}$, $\hat{g}$ are isomorphisms in $C(\mathscr{A})$ with $\hat{g} = \hat{h}^{-1}$. Put

$$\hat{u} = \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} : X^\bullet \to Y^\bullet \oplus C(\mathrm{id}_X),$$

$$\hat{v} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & u \end{bmatrix} : Y^\bullet \oplus C(\mathrm{id}_X) \to C(u).$$

Then the following diagram in $C(\mathscr{A})$ commutes

$$\begin{array}{ccccccc} T^{-1}(C(u)) & \xrightarrow{-T^{-1}w} & X^\bullet & \xrightarrow{\hat{u}} & Y^\bullet \oplus C(\mathrm{id}_X) & \xrightarrow{\hat{v}} & C(u) \\ \| & & \| & & \downarrow \hat{h} & & \| \\ T^{-1}(C(u)) & \xrightarrow{-T^{-1}w} & X^\bullet & \xrightarrow{u'} & C(-T^{-1}w) & \xrightarrow{v'} & C(u), \end{array}$$

where $u' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : X^\bullet \to C(-T^{-1}w)$, $v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : C(-T^{-1}w) \to C(u)$. Thus we get the

following commutative diagram in $K(\mathscr{A})$

$$\begin{array}{ccccccc} T^{-1}(C(u)) & \xrightarrow{-T^{-1}w} & X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{v} & C(u) \\ \| & & \| & & \downarrow h & & \| \\ T^{-1}(C(u)) & \xrightarrow{-T^{-1}w} & X^\bullet & \xrightarrow{u'} & C(-T^{-1}w) & \xrightarrow{v'} & C(u)\ , \end{array}$$

where $h = {}^t[0\ \ 1\ \ 0]$ is an isomorphism in $K(\mathscr{A})$.

Proposition 5.5. *For any commutative square in* $K(\mathscr{A})$

$$\begin{array}{ccc} X^\bullet & \xrightarrow{u} & Y^\bullet \\ f\downarrow & & \downarrow g \\ X'^\bullet & \xrightarrow{u'} & Y'^\bullet \end{array}$$

there exists $h \in K(\mathscr{A})(C(u), C(u'))$ *such that*

$$(f, g, h) : (X^\bullet, Y^\bullet, C(u), u, v, w) \to (X'^\bullet, Y'^\bullet, C(u'), u', v', w')$$

is a homomorphism of mapping cylinders.

Proof. Let $\hat{u} = {}^t[u\ \ 0\ \ 1] : X^\bullet \to Y^\bullet \oplus C(\mathrm{id}_X)$. Since $u'f - gu \in \mathrm{Htp}(X^\bullet, Y'^\bullet)$, by Proposition 2.1 there exists $[a\ \ b] : C(\mathrm{id}_X) \to Y'^\bullet$ such that $u'f = gu + b$. Put

$$\hat{v} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : Y^\bullet \oplus C(\mathrm{id}_X) \to C(u) \oplus C(\mathrm{id}_X),$$

$$\hat{w} = [1\ \ 0\ \ 0\ \ 0] : C(u) \oplus C(\mathrm{id}_X) \to T X^\bullet,$$

$$\hat{g} = [g\ \ a\ \ b] : Y^\bullet \oplus C(\mathrm{id}_X) \to Y'^\bullet,$$

$$\hat{h} = \begin{bmatrix} Tf & 0 & 0 & 0 \\ -a & g & a & b \end{bmatrix} : C(u) \oplus C(\mathrm{id}_X) \to C(u').$$

Then the following diagram in $C(\mathscr{A})$ commutes

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{\hat{u}} & Y^\bullet \oplus C(\mathrm{id}_X) & \xrightarrow{\hat{v}} & C(u) \oplus C(\mathrm{id}_X) & \xrightarrow{\hat{w}} & T X^\bullet \\ f\downarrow & & \downarrow \hat{g} & & \downarrow \hat{h} & & \downarrow Tf \\ X'^\bullet & \xrightarrow{u'} & Y'^\bullet & \xrightarrow{v'} & C(u') & \xrightarrow{w'} & T X'^\bullet\ , \end{array}$$

where $v' = {}^t[0 \ \ 1] : Y'^{\bullet} \to C(u')$, $w' = [1 \ \ 0] : C(u') \to TX'^{\bullet}$. Thus we get the following commutative diagram in $K(\mathscr{A})$

$$\begin{array}{ccccccc} X^{\bullet} & \xrightarrow{u} & Y^{\bullet} & \xrightarrow{v} & C(u) & \xrightarrow{w} & TX^{\bullet} \\ f\downarrow & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X'^{\bullet} & \xrightarrow{u'} & Y'^{\bullet} & \xrightarrow{v'} & C(u') & \xrightarrow{w'} & TX'^{\bullet}, \end{array}$$

where $v = {}^t[0 \ \ 1] : Y^{\bullet} \to C(u)$, $w = [1 \ \ 0] : C(u) \to TX^{\bullet}$ and $h = \begin{bmatrix} Tf & 0 \\ -a & b \end{bmatrix}$.

Proposition 5.6. *For two consecutive morphisms* $u : X^{\bullet} \to Y^{\bullet}$, $v : Y^{\bullet} \to Z^{\bullet}$ *in* $C(\mathscr{A})$, *take the mapping cylinders*

$$(X^{\bullet}, Y^{\bullet}, C(u), u, i, \cdot),\ (X^{\bullet}, Z^{\bullet}, C(vu), vu, j, \cdot),\ (Y^{\bullet}, Z^{\bullet}, C(v), v, \cdot, k)$$

and put $f = \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix} : C(u) \to C(vu)$, $g = \begin{bmatrix} Tu & 0 \\ 0 & 1 \end{bmatrix} : C(vu) \to C(v)$. *Then the following hold.*

(1) *The following diagram in* $C(\mathscr{A})$ *commutes*

$$\begin{array}{ccccccc} X^{\bullet} & \xrightarrow{u} & Y^{\bullet} & \xrightarrow{i} & C(u) & \to & TX^{\bullet} \\ \| & & \downarrow v & & \downarrow f & & \| \\ X^{\bullet} & \xrightarrow{vu} & Z^{\bullet} & \xrightarrow{j} & C(vu) & \to & TX^{\bullet} \\ u\downarrow & & \| & & \downarrow g & & \downarrow Tu \\ Y^{\bullet} & \xrightarrow{v} & Z^{\bullet} & \to & C(v) & \xrightarrow{k} & TY^{\bullet} \\ i\downarrow & & \downarrow j & & \| & & \downarrow Ti \\ C(u) & \xrightarrow{f} & C(vu) & \xrightarrow{g} & C(v) & \xrightarrow{(Ti)k} & T(C(u)). \end{array}$$

(2) $(C(u), C(vu), C(v), f, g, T(i)k)$ *is isomorphic to the mapping cylinder of* f.

Proof. (1) Straightforward.

(2) Put

$$\phi = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -Tu \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} : C(v) \oplus C(\mathrm{id}_{TX}) \to C(f),$$

$$\psi = \begin{bmatrix} 0 & 1 & Tu & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} : C(f) \to C(v) \oplus C(\mathrm{id}_{TX}).$$

Then ϕ, ψ are isomorphisms in $C(\mathscr{A})$ with $\psi = \phi^{-1}$. Put

$$\hat{g} = \begin{bmatrix} Tu & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} : C(vu) \to C(v) \oplus C(\mathrm{id}_{TX}),$$

$$\hat{h} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -Tu \end{bmatrix} : C(v) \oplus C(\mathrm{id}_{TX}) \to T(C(u)).$$

Then the following diagram in $C(\mathscr{A})$ commutes

$$\begin{array}{ccccccc} C(u) & \xrightarrow{f} & C(vu) & \xrightarrow{\hat{g}} & C(v) \oplus C(\mathrm{id}_{TX}) & \xrightarrow{\hat{h}} & T(C(u)) \\ \| & & \| & & \downarrow \phi & & \| \\ C(u) & \xrightarrow{f} & C(vu) & \xrightarrow{g'} & C(f) & \xrightarrow{h'} & T(C(u)) \end{array},$$

where $g' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} : C(vu) \to C(f)$, $h' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} : C(f) \to T(C(u))$. Thus we get the following commutative diagram in $K(\mathscr{A})$

$$\begin{array}{ccccccc} C(u) & \xrightarrow{f} & C(vu) & \xrightarrow{g} & C(v) & \xrightarrow{(Ti)k} & T(C(u)) \\ \| & & \| & & \downarrow \bar{\phi} & & \| \\ C(u) & \xrightarrow{f} & C(vu) & \xrightarrow{g'} & C(f) & \xrightarrow{h'} & T(C(u)) \end{array},$$

where $\overline{\phi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ is an isomorphism in $K(\mathscr{A})$.

§6. Triangulated categories

Throughout this section, unless otherwise stated, functors are covariant functors.

Definition 6.1. A triangulated category is an additive category $\mathcal{K}$, together with (1) an autofunctor $T : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$, called the translation, and (2) a collection of cylinders (X, Y, Z, u, v, w), called (distinguished) triangles. This data is subject to the following four axioms:

(TR1) (1) Every cylinder (X, Y, Z, u, v, w) which is isomorphic to a triangle is a triangle.
(2) Every morphism $u : X \to Y$ is embedded in a triangle (X, Y, Z, u, v, w).
(3) The cylinder $(X, X, 0, \mathrm{id}_X, 0, 0)$ is a triangle for all $X \in \mathrm{Ob}(\mathcal{K})$.

(TR2) A cylinder (X, Y, Z, u, v, w) is a triangle if and only if $(Y, Z, TX, v, w, -Tu)$ is a triangle.

(TR3) For any triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and morphisms $f : X \to X'$, $g : Y \to Y'$ with $gu = u'f$, there exists $h : Z \to Z'$ such that (f, g, h) is a homomorphism of triangles.

(TR4) (Octahedral axiom) For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed u, vu and v in triangles

$$(X, Y, Z', u, i, \cdot),\ (X, Z, Y', vu, j, \cdot)\ \text{ and }\ (Y, Z, X', v, \cdot, k),$$

respectively, then there exist morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commute

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \to & TX \\
\| & & \downarrow v & & \downarrow f & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{j} & Y' & \to & TX \\
u \downarrow & & \| & & \downarrow g & & \downarrow Tu \\
Y & \xrightarrow{v} & Z & \to & X' & \xrightarrow{k} & TY \\
i \downarrow & & \downarrow j & & \| & & \downarrow Ti \\
Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \to & TZ'
\end{array}$$

and the bottom row is a triangle.

Remark 6.1. (TR4) is equivalent to the following.

(TR4)' For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed u, vu and v in triangles

$$(X, Y, Z', u, i, \cdot), \ (X, Z, Y', vu, j, \cdot) \ \text{and} \ (Y, Z, X', v, \cdot, k),$$

respectively, then there exists a commutative diagram

$$\begin{array}{ccccccc}
 & & T^{-1}X' & = & T^{-1}X' & & \\
 & & {\scriptstyle -T^{-1}k}\downarrow & & \downarrow & & \\
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \to & TX \\
\| & & \downarrow{\scriptstyle v} & & \downarrow & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{j} & Y' & \to & TX \\
 & & \downarrow & & \downarrow & & \\
 & & X' & = & X' & &
\end{array}$$

with the rows and the columns being triangles.

Remark 6.2. If $\mathscr{K}$ is a triangulated category, then the opposite category $\mathscr{K}^{\mathrm{op}}$ is also a triangulated category with the translation T^{-1}.

Proposition 6.1. *Let* $\mathscr{A}$ *be an abelian category. Then, for* $* = +, -, \mathrm{b}, (+, \mathrm{b}), (-, \mathrm{b})$ *or nothing, the following hold.*

(1) $K^*(\mathscr{A})$ *is a triangulated category.*

(2) *If* $\mathscr{L}$ *is a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *containing zero objects and closed under finite diret sums, then* $K^*(\mathscr{L})$ *is a full triangulated subcategory of* $K^*(\mathscr{A})$.

Proof. (1) See Section 5.

(2) It is obvious that $K^*(\mathscr{L})$ is stable under the translation T. Since $\mathscr{L}$ is closed under finite direct sums, $K^*(\mathscr{L})$ is closed under mapping cones.

Throughout the rest of this section, we work over a triangulated category $\mathscr{K}$. However, except Lemma 6.12, we will not need the octahedral axiom.

Lemma 6.2. *If* (X, Y, Z, u, v, w) *is a triangle, then*

$$(X, Y, Z, -u, -v, w), \ (X, Y, Z, -u, v, -w) \ \text{and} \ (X, Y, Z, u, -v, -w)$$

are triangles.

Proof. According to (TR2), it suffices to prove one of them is a triangle. Since we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \| & & \downarrow{\scriptstyle -1} & & \| & & \| \\ X & \xrightarrow{-u} & Y & \xrightarrow{-v} & Z & \xrightarrow{w} & TX, \end{array}$$

$(X, Y, Z, -u, -v, w)$ is a triangle.

Lemma 6.3. *Let* (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') *be triangles. Then the following hold.*

(1) *For any* $f : X \to X'$ *and* $h : Z \to Z'$ *with* $T(f)w = w'h$, *there exists* $g : Y \to Y'$ *such that* (f, g, h) *is a homomorphism of triangles*.

(2) *For any* $g : Y \to Y'$ *and* $h : Z \to Z'$ *with* $hv = v'g$, *there exists* $f : X \to X'$ *such that* (f, g, h) *is a homomorphism of triangles*.

Proof. (1) Since $f \circ (-T^{-1}w) = -T^{-1}(T(f) \circ w) = -T^{-1}(w' \circ h) = -T^{-1}w' \circ T^{-1}h$, and since by (TR2)

$$(T^{-1}Z, X, Y, -T^{-1}w, u, v), \quad (T^{-1}Z', X', Y', -T^{-1}w', u', v')$$

are triangles, (TR3) applies.

(2) Similar to (1).

Lemma 6.4. *If* (X, Y, Z, u, v, w) *is a triangle, then* $vu = 0$, $wv = 0$ *and* $T(w)u = 0$.

Proof. According to (TR2), it suffices to show $vu = 0$. By (TR2) and (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X & \rightarrow & 0 & \rightarrow & TX \\ \| & & \downarrow{\scriptstyle u} & & \downarrow & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX. \end{array}$$

Definition 6.2. Let $\mathscr{A}$ be an abelian category. An additive functor $H : \mathscr{K} \to \mathscr{A}$ is called a cohomological functor if, for any triangle (X, Y, Z, u, v, w), the induced sequence

$$\cdots \to H(T^nX) \to H(T^nY) \to H(T^nZ) \to H(T^{n+1}X) \to \cdots$$

is exact. If $H : \mathscr{K} \to \mathscr{A}$ is a cohomological functor, we set $H^n = H \circ T^n$ for all $n \in \mathbb{Z}$ and define an additive functor

$$H^\bullet : \mathscr{K} \to \mathscr{A}^{\mathbb{Z}}, X \mapsto \{H^n(\mathrm{X})\}_{n \in \mathbb{Z}}.$$

A contravariant cohomological functor $H : \mathscr{K} \to \mathscr{A}$ is defined as a covariant cohomological functor $H : \mathscr{K}^{\mathrm{op}} \to \mathscr{A}$. In this case, we set $H^n = H \circ T^{-n} : \mathscr{K} \to \mathscr{A}$ for $n \in \mathbb{Z}$.

Proposition 6.5. *For any* $W \in \mathrm{Ob}(\mathscr{K})$ *the following hold.*

(1) $\mathscr{K}(W, -) : \mathscr{K} \to \mathrm{Mod}\,\mathbb{Z}$ *is a covariant cohomological functor.*

(2) $\mathscr{K}(-, W) : \mathscr{K} \to \mathrm{Mod}\,\mathbb{Z}$ *is a contravariant cohomological functor.*

Proof. (1) Let (X, Y, Z, u, v, w) be a triangle. Then, since by Lemma 6.5 $vu = 0$, we have $\mathscr{K}(W, v) \circ \mathscr{K}(W, u) = 0$. Conversely, let $g \in \mathscr{K}(W, Y)$ with $\mathscr{K}(W, v)(g) = vg = 0$. Then by Lemma 6.3 there exists $f \in \mathscr{K}(W, Y)$ which makes the following diagram commute

$$\begin{array}{ccccccc} W & \xrightarrow{1} & W & \to & 0 & \to & TW \\ f\downarrow & & \downarrow g & & \downarrow & & \downarrow Tf \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX. \end{array}$$

Thus $g = \mathscr{K}(W, u)(f)$ and the sequence

$$\mathscr{K}(W, X) \to \mathscr{K}(W, Y) \to \mathscr{K}(W, Z)$$

is exact. It follows by (TR2) that $\mathscr{K}(W, -)$ is a cohomological functor.

(2) Dual of (1).

Proposition 6.6. *For any homomorphism of triangles*

$$(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w'),$$

if two of f, g and h are isomorphisms, then the rest is also an isomorphism.

Proof. According to (TR2), it is enough to deal with the case where f, g are isomorphisms. By Proposition 6.5 we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \mathcal{K}(TY', -) & \to & \mathcal{K}(TX', -) & \to & \mathcal{K}(Z', -) & \to & \mathcal{K}(Y', -) & \to & \mathcal{K}(X', -) \\ {\scriptstyle \mathcal{K}(Tg,-)}\downarrow & & \downarrow{\scriptstyle \mathcal{K}(Tf,-)} & & \downarrow{\scriptstyle \mathcal{K}(h,-)} & & \downarrow{\scriptstyle \mathcal{K}(g,-)} & & \downarrow{\scriptstyle \mathcal{K}(f,-)} \\ \mathcal{K}(TY, -) & \to & \mathcal{K}(TX, -) & \to & \mathcal{K}(Z, -) & \to & \mathcal{K}(Y, -) & \to & \mathcal{K}(X, -). \end{array}$$

Thus, since by five-lemma $\mathcal{K}(h, -)$ is an isomorphism, it follows by Yoneda lemma that h is an isomorphism.

Corollary 6.7. *For any morphism* $u \in \mathcal{K}(X, Y)$, *the triangle* (X, Y, Z, u, v, w) *is unique up to isomorphisms*.

Proof. Let (X, Y, Z', u, v', w') be another triangle. Then by (TR3) there exists $h : Z \to Z'$ which makes the following diagram commute

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \| & & \| & & \downarrow{\scriptstyle h} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX. \end{array}$$

It follows by Proposition 6.6 that h is an isomorphism.

Definition 6.3. Let (X, Y, Z, u, v, w) be a triangle. Then by Corollary 6.7 Z is uniquely determined by u up to isomorphisms. So, sometimes we call Z the mapping cone of u and denote it by $C(u)$.

Lemma 6.8. *For a triangle* (X, Y, Z, u, v, w) *the following are equivalent*.

(1) u *is a section, i.e.*, $\mathcal{K}(u, X)$ *is surjective*.

(2) v *is a retraction, i.e.*, $\mathcal{K}(Z, v)$ *is surjective*.

(3) $w = 0$.

Proof. (1) $\Rightarrow$ (3). We have the following commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\ X & \xrightarrow{1} & X & \to & 0 & \to & TX. \end{array}$$

(2) ⇒ (3). Dual of (1) ⇒ (3).
(3) ⇒ (1) and (2). By Proposition 6.5.

Lemma 6.9. *For a triangle* (X, Y, Z, u, v, w) *the following are equivalent.*
(1) *u is an isomorphism.*
(2) $Z = 0$.

Proof. (1) ⇒ (2). Since by (TR2) and (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X & \to & 0 & \to & TX \\ \| & & \downarrow u & & \downarrow & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX, \end{array}$$

by Proposition 6.6 $Z = 0$.
(2) ⇒ (1). By Lemma 6.8 u is a section. Also, according to (TR2), again by Lemma 6.8 u is a retraction.

Proposition 6.10. *Let* Λ *be a set and* $\{(X_\lambda, Y_\lambda, Z_\lambda, u_\lambda, v_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ *a family of cylinders. Then the following hold.*
(1) *Assume the constant functor* $\mathcal{K} \to \mathcal{K}^\Lambda$ *has a right adjoint* $\prod_{\lambda \in \Lambda} : \mathcal{K}^\Lambda \to \mathcal{K}$. *Then the direct product of cylinders*

$$(\prod X_\lambda, \prod Y_\lambda, \prod Z_\lambda, \prod u_\lambda, \prod v_\lambda, \prod w_\lambda)$$

is a triangle if and only if every cylinder $(X_\lambda, Y_\lambda, Z_\lambda, u_\lambda, v_\lambda, w_\lambda)$ *is a triangle.*
(2) *Assume the constant functor* $\mathcal{K} \to \mathcal{K}^\Lambda$ *has a left adjoint* $\oplus_{\lambda \in \Lambda} : \mathcal{K}^\Lambda \to \mathcal{K}$. *Then the direct sum of cylinders*

$$(\oplus X_\lambda, \oplus Y_\lambda, \oplus Z_\lambda, \oplus u_\lambda, \oplus v_\lambda, \oplus w_\lambda)$$

is a triangle if and only if every cylinder $(X_\lambda, Y_\lambda, Z_\lambda, u_\lambda, v_\lambda, w_\lambda)$ *is a triangle.*

Proof. (1) Note first that there exists a natural isomorphism $T(\prod W_\lambda) \xrightarrow{\sim} \prod TW_\lambda$ for a family of objects $\{W_\lambda\}_{\lambda \in \Lambda}$. For each $\mu \in \Lambda$, we denote by

$$p_\mu : \prod X_\lambda \to X_\mu, \quad q_\mu : \prod Y_\lambda \to Y_\mu \quad \text{and} \quad r_\mu : \prod Z_\lambda \to Z_\mu$$

projections.

"If" part. By (TR1) we have a triangle of the form $(\prod X_\lambda, \prod Y_\lambda, Z, \prod u_\lambda, v, w)$. Then, for each $\mu \in \Lambda$, by (TR3) we have a homomorphism of triangles

$$\begin{array}{ccccccc} \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{v} & Z & \xrightarrow{w} & \prod TX_\lambda \\ p_\mu \downarrow & & \downarrow q_\mu & & \downarrow h_\mu & & \downarrow Tp_\mu \\ X_\mu & \xrightarrow{u_\mu} & Y_\mu & \xrightarrow{v_\mu} & Z_\mu & \xrightarrow{w_\mu} & TX_\mu . \end{array}$$

Thus we get a commutative diagram

$$\begin{array}{ccccccc} \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{v} & Z & \xrightarrow{w} & \prod TX_\lambda \\ \| & & \| & & \downarrow h & & \| \\ \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{\prod v_\lambda} & \prod Z_\lambda & \xrightarrow{\prod w_\lambda} & \prod TX_\lambda . \end{array}$$

It suffices to show that h is an isomorphism. We have a commutative diagram of functors

$$\begin{array}{ccccccccc} \mathcal{K}(-, \prod X_\lambda) & \to & \mathcal{K}(-, \prod Y_\lambda) & \to & \mathcal{K}(-, Z) & \to & \mathcal{K}(-, \prod TX_\lambda) & \to & \mathcal{K}(-, \prod TY_\lambda) \\ \| & & \| & & \downarrow \mathcal{K}(-, h) & & \| & & \| \\ \mathcal{K}(-, \prod X_\lambda) & \to & \mathcal{K}(-, \prod Y_\lambda) & \to & \mathcal{K}(-, \prod Z_\lambda) & \to & \mathcal{K}(-, \prod TX_\lambda) & \to & \mathcal{K}(-, \prod TY_\lambda) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod \mathcal{K}(-, X_\lambda) & \to & \prod \mathcal{K}(-, Y_\lambda) & \to & \prod \mathcal{K}(-, Z_\lambda) & \to & \prod \mathcal{K}(-, TX_\lambda) & \to & \prod \mathcal{K}(-, TY_\lambda) . \end{array}$$

By Proposition 6.5 the top and the bottom rows are exact, so is the middle one. Thus by five-lemma $\mathcal{K}(-, h)$ is an isomorphism, so is h by Yoneda lemma.

"Only if" part. By (TR1) we have a family of triangles $\{(X_\lambda, Y_\lambda, Z_\lambda', u_\lambda, v_\lambda', w_\lambda')\}_{\lambda \in \Lambda}$. Since by the "if" part we have a triangle of the form

$$(\prod X_\lambda, \prod Y_\lambda, \prod Z_\lambda', \prod u_\lambda, \prod v_\lambda', \prod w_\lambda'),$$

by (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{\prod v_\lambda} & \prod Z_\lambda & \xrightarrow{\prod w_\lambda} & \prod TX_\lambda \\ \| & & \| & & \downarrow h & & \| \\ \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{\prod v_\lambda'} & \prod Z_\lambda' & \xrightarrow{\prod w_\lambda'} & \prod TX_\lambda . \end{array}$$

For each $\mu \in \Lambda$, we denote by $r_\mu' : \prod Z_\lambda' \to Z_\mu'$ the projection. Also, for each $\nu \in \Lambda$, there exists $i_\nu : Z_\nu \to \prod Z_\lambda$ such that

$$r_\mu \circ i_\nu = \begin{cases} \mathrm{id}_{Z_\nu} & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases}.$$

Thus, by setting

$$h_\mu : Z_\mu \xrightarrow{i_\mu} \prod Z_\lambda \xrightarrow{h} \prod Z_\lambda' \xrightarrow{r'_\mu} Z_\mu'$$

for each $\mu \in \Lambda$, we get a commutative diagram

$$\begin{array}{ccccccc} \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{\prod v_\lambda} & \prod Z_\lambda & \xrightarrow{\prod w_\lambda} & \prod TX_\lambda \\ \| & & \| & & \downarrow \prod h_\lambda & & \| \\ \prod X_\lambda & \xrightarrow{\prod u_\lambda} & \prod Y_\lambda & \xrightarrow{\prod v'_\lambda} & \prod Z_\lambda' & \xrightarrow{\prod w'_\lambda} & \prod TX_\lambda . \end{array}$$

By Proposition 6.6 $\prod h_\lambda : \prod Z_\lambda \to \prod Z_\lambda'$ is an isomorphism and, for each $\lambda \in \Lambda$, we get an isomorphism of cylinders

$$\begin{array}{ccccccc} X_\lambda & \xrightarrow{u_\lambda} & Y_\lambda & \xrightarrow{v_\lambda} & Z_\lambda & \xrightarrow{w_\lambda} & TX_\lambda \\ \| & & \| & & \downarrow h_\lambda & & \| \\ X_\lambda & \xrightarrow{u_\lambda} & Y_\lambda & \xrightarrow{v'_\lambda} & Z_\lambda' & \xrightarrow{w'_\lambda} & TX_\lambda . \end{array}$$

(2) Dual of (1).

Corollary 6.11. *A triangle* $(X, Y, Z, u, v, 0)$ *decomposes into a direct sum*

$$(X, Z \oplus X, Z, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1\ \ 0], 0).$$

Proof. Since we have triangles $(X, X, 0, \mathrm{id}_X, 0, 0)$, $(0, Z, Z, 0, \mathrm{id}_Z, 0)$, by Proposition 6.10 $(X, Z \oplus X, Z, {}^t[0\ \ 1], [1\ \ 0], 0)$ is a triangle. Also, since by Lemma 6.8 there exists $h : Z \to Y$ with $vh = \mathrm{id}_Z$. we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u'} & Z \oplus X & \xrightarrow{v'} & Z & \xrightarrow{0} & TX \\ \| & & \downarrow [u\ \ h] & & \| & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{0} & TX , \end{array}$$

where $u' = {}^t[0\ \ 1]$, $v' = [1\ \ 0]$. Thus by Proposition 6.6 the assertion follows.

Lemma 6.12. *For any homomorphism of triangles*

$$(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$$

with h an isomorphism, there exists $g' : Y \to Y'$ such that

$$\left(X,\ Y \oplus X',\ Y',\ \begin{bmatrix} u \\ f \end{bmatrix},\ [g'\ \ -u'],\ wh^{-1}v'\right)$$

is a triangle and (f, g', h) is a homomorphism of triangles.

Proof. Put $\omega = wh^{-1}v'$. We have an isomorphism of cylinders

$$\begin{array}{ccccccc} Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' & \xrightarrow{-Tu'} & TY' \\ \| & & \downarrow h^{-1} & & \downarrow -1 & & \| \\ Y' & \xrightarrow{h^{-1}v'} & Z & \xrightarrow{-w'h} & TX' & \xrightarrow{Tu'} & TY'. \end{array}$$

Since the top row is a triangle, so is the bottom one. Note that by Lemma 6.4 $w'hv = T(f)wv = 0$. Thus by (TR4) we have a commutative diagram

$$\begin{array}{ccccccc} Y' & \xrightarrow{h^{-1}v'} & Z & \xrightarrow{-w'h} & TX' & \xrightarrow{Tu'} & TY' \\ \| & & \downarrow w & & \downarrow \gamma & & \| \\ Y' & \xrightarrow{\omega} & TX & \xrightarrow{\phi} & C(\omega) & \xrightarrow{\psi} & TY' \\ h^{-1}v' \downarrow & & \| & & \downarrow \delta & & \downarrow T(h^{-1}v') \\ Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY & \xrightarrow{-Tv} & TZ \\ w'h \downarrow & & \downarrow \phi & & \| & & \downarrow T(w'h) \\ TX' & \xrightarrow{\gamma} & C(\omega) & \xrightarrow{\delta} & TY & \xrightarrow{0} & T^2X' \end{array}$$

with the rows being triangles. Thus by Corollary 6.11 we have an isomorphism of triangles

$$\begin{array}{ccccccc} TX' & \xrightarrow{\mu} & TY \oplus TX' & \xrightarrow{\varepsilon} & TY & \xrightarrow{0} & T^2X' \\ \| & & \downarrow [\eta\ \ \gamma] & & \| & & \| \\ TX' & \xrightarrow{\gamma} & C(\omega) & \xrightarrow{\delta} & TY & \xrightarrow{0} & T^2X', \end{array}$$

where $\mu = {}^t[0\ \ 1]$, $\varepsilon = [1\ \ 0]$. Note that $[\eta\ \ \gamma]^{-1}$ is of the form ${}^t[\delta\ \ \theta]$, where $\delta\eta = \mathrm{id}_{TY}$, $\delta\gamma = 0$, $\theta\eta = 0$, $\theta\gamma = \mathrm{id}_{TX'}$ and $\eta\delta + \gamma\theta = \mathrm{id}_{C(\omega)}$. Also, $\delta\phi = -Tu$ and $\psi\gamma = Tu'$. Put $\phi' = {}^t[-Tu\ \ \ \theta\phi]$ and $\psi' = [\psi\eta\ \ \ Tu']$. Then we have an isomorphism of cylinders

$$\begin{array}{ccccccc} Y' & \xrightarrow{\omega} & TX & \xrightarrow{\phi'} & TY \oplus TX' & \xrightarrow{\psi'} & TY' \\ \| & & \| & & \downarrow [\eta\ \gamma] & & \| \\ Y' & \xrightarrow{\omega} & TX & \xrightarrow{\phi} & C(\omega) & \xrightarrow{\psi} & TY' \end{array}$$

and the top row is also a triangle. Put $f' = -T^{-1}(\theta\phi)$ and $g'' = -T^{-1}(\psi\eta)$. Then $-T^{-1}(\phi') = {}^t[u\ \ \ f']$, $-T^{-1}(\psi') = [g''\ \ \ -u']$ and by (TR2)

$$(X, Y \oplus X', Y', \begin{bmatrix} u \\ f' \end{bmatrix}, [g''\ \ \ -u'], \omega)$$

is a triangle. In particular, by Lemma 6.4 $u'f' = g''u$. Since $\delta\eta = \mathrm{id}_{TY}$, we have

$$\begin{aligned} T(v')T(g'') &= -T(v')\psi\eta \\ &= -T(h)T(h^{-1}v')\psi\eta \\ &= T(h)T(v)\delta\eta \\ &= T(h)T(v), \end{aligned}$$

so that $v'g'' = hv$. Also, since $\theta\gamma = \mathrm{id}_{TX'}$, we have

$$\begin{aligned} T(f')w &= -\theta\phi w \\ &= \theta\gamma w'h \\ &= w'h. \end{aligned}$$

Thus (f', g'', h) is a homomorphism of triangles, so is $(f' - f, g'' - g, 0)$. Hence by Proposition 6.5 there exists $\varphi : Y \to X'$ such that $f' - f = u'\varphi$. Put $g' = g'' - u'\varphi$. Then we have an isomorphism of cylinders

$$\begin{array}{ccccccc} X & \xrightarrow{\hat{f}} & Y \oplus X' & \xrightarrow{\hat{g}'} & Y' & \xrightarrow{\omega} & TX \\ \| & & \downarrow \hat{\varphi} & & \| & & \| \\ X & \xrightarrow{\hat{f}'} & Y \oplus X' & \xrightarrow{\hat{g}''} & Y' & \xrightarrow{\omega} & TX, \end{array}$$

where $\hat{f} = {}^t[u\ \ \ f]$, $\hat{g}' = [g'\ \ \ -u']$, $\hat{f}' = {}^t[u\ \ \ f']$, $\hat{g}'' = [g''\ \ \ -u']$ and $\hat{\varphi} = \begin{bmatrix} 1 & 0 \\ \varphi & 1 \end{bmatrix}$. Thus the top row is a triangle. Since $\hat{g}'\hat{f} = 0$, $g'u = u'f$. Also, $v'g' = v'(g'' - u'\varphi) = v'g'' = hv$ and $T(f)w = (T(f') - T(\varphi)T(u))w = T(f')w = w'h$. Thus (f, g', h) is a homomorphism of triangles.

Lemma 6.13. *Let* (X, Y, Z, u, v, w) *be a triangle and* $\varphi : Y \to Y$ *with* $u = \varphi u$. *Then* ${}^t[v \quad \varphi] : Y \to Z \oplus Y$ *is a section*.

Proof. By (TR2) and (TR3) we have a homomorphism of triangles

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY \\ \varphi\downarrow & & \downarrow & & \| & & \downarrow T\varphi \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY. \end{array}$$

Since by Lemma 6.4 $T(u)w = 0$, by Lemma 6.12 we have a triangle of the form

$$(Y, Z \oplus Y, Z, \begin{bmatrix} v \\ \varphi \end{bmatrix}, \cdot, 0).$$

It follows by Lemma 6.8 that ${}^t[v \quad \varphi] : Y \to Z \oplus Y$ is a section.

Lemma 6.14. *Let* $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ *be a homomorphism of triangles. Then for any* $g' : Y \to Y'$ *the following are equivalent*.

(1) (f, g', h) *is also a homomorphism of triangles*.

(2) *There exists* $\phi : Y \to X'$ *such that* $g' = g + u'\phi$ *and* $u'\phi u = 0$.

(3) *There exists* $\psi : Z \to Y'$ *such that* $g' = g + \psi v$ *and* $v'\psi v = 0$.

Proof. (1) $\Rightarrow$ (2). Since $(0, g' - g, 0)$ is a homomorphism of triangles, $v'(g' - g) = 0$ and by Proposition 6.5 there exists $\phi : Y \to X'$ such that $g' - g = u'\phi$. Then $u'f = g'u = (g + u'\phi)u = gu + u'\phi u = u'f + u'\phi u$, so that $u'\phi u = 0$.

(2) $\Rightarrow$ (1). We have $g'u = (g + u'\phi)u = gu = u'f$. Also, since by Lemma 6.4 $v'u' = 0$, $v'g' = v'(g + u'\phi) = v'g = hv$.

(1) $\Leftrightarrow$ (3). Dual of (1) $\Leftrightarrow$ (2).

Lemma 6.15. *Let* $\mathscr{A}$ *be an abelian category and* $\mathscr{I}$ *(resp.* $\mathscr{P}$*) the collection of injective (resp. projective) objects of* $\mathscr{A}$. *Then for any quasi-isomorphism* $u : X^\bullet \to Y^\bullet$ *the following hold.*

(1) $K(\mathscr{A})(u, I^\bullet)$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

(2) $K(\mathscr{A})(P^\bullet, u)$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$.

Proof. (1) Let $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. Since $(X^\bullet, Y^\bullet, C(u), u, \cdot, \cdot)$ is a triangle in $K(\mathscr{A})$, by Proposition 6.5 we have an exact sequence

$$\cdots \to K(\mathscr{A})(C(u),\ I^\bullet) \to K(\mathscr{A})(Y^\bullet,\ I^\bullet) \to K(\mathscr{A})(X^\bullet,\ I^\bullet) \to K(\mathscr{A})(T^{-1}C(u),\ I^\bullet) \to \cdots .$$

Also, since $C(u)$ is acyclic, by Lemma 4.4 we have

$$K(\mathscr{A})(C(u),\ I^\bullet) = K(\mathscr{A})(T^{-1}C(u),\ I^\bullet) = 0.$$

Thus $K(\mathscr{A})(u,\ I^\bullet)$ is an isomorphism.

(2) Dual of (1).

§7. Épaisse subcategories

Throughout this section, $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{G}$ are triangulated categories. Unless otherwise stated, functors are covariant functors.

Definition 7.1. An épaisse subcategory $\mathcal{U}$ of a triangulated category $\mathcal{K}$ is a full triangulated subcategory of $\mathcal{K}$ such that if $u \in \mathcal{K}(X, Y)$ factors through (an object of) $\mathcal{U}$ and is embedded in a triangle $(X, Y, Z, u, \cdot, \cdot)$ in $\mathcal{K}$ with $Z \in \mathrm{Ob}(\mathcal{U})$ then $X, Y \in \mathrm{Ob}(\mathcal{U})$.

Proposition 7.1. *For a full triangulated subcategory* $\mathcal{U}$ *of* $\mathcal{K}$*, the following are equivalent*.

(1) $\mathcal{U}$ *is an épaisse subcategory of* $\mathcal{K}$.

(2) $\mathcal{U}$ *is closed underisomorphism classes and taking direct summands.*

Proof. (1) ⇒ (2). For any isomorphism $u : X \to Y$ with $Y \in \mathrm{Ob}(\mathcal{U})$, since by Lemma 6.9 we have a triangle of the form $(X, Y, 0, u, 0, 0)$, $X \in \mathrm{Ob}(\mathcal{U})$. Note that zero morphisms factor through $\mathcal{U}$. Thus, for any $X, Y \in \mathrm{Ob}(\mathcal{K})$, since by Corollary 6.11 we have a triangle of the form $(T^{-1}X, Y, X \oplus Y, 0, \cdot, \cdot)$, $X \oplus Y \in \mathrm{Ob}(\mathcal{U})$ implies $T^{-1}X, Y \in \mathrm{Ob}(\mathcal{U})$.

(2) ⇒ (1). Let $(X, Y, Z, u, \cdot, \cdot)$ be a triangle in $\mathcal{K}$ such that $Z \in \mathrm{Ob}(\mathcal{U})$ and u factors through $Y' \in \mathrm{Ob}(\mathcal{U})$. We claim $X, Y \in \mathrm{Ob}(\mathcal{U})$. Let $u' : X \to Y'$, $u'' : Y' \to Y$ with $u = u''u'$. Then by (TR3) we have a homomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u'} & Y' & \to & Z' & \to & TX \\ \| & & \downarrow{\scriptstyle u''} & & \downarrow & & \| \\ X & \xrightarrow{u} & Y & \to & Z & \to & TX. \end{array}$$

Thus by Lemma 6.12 we have a triangle of the form $(Y', Z' \oplus Y, Z, \cdot, \cdot, \cdot)$. Thus $Z' \oplus Y \in \mathrm{Ob}(\mathcal{U})$, so that $Y \in \mathrm{Ob}(\mathcal{U})$. It then follows that $X \in \mathrm{Ob}(\mathcal{U})$.

Definition 7.2. For an épaisse subcategory $\mathcal{U}$ of $\mathcal{K}$, we denote by $\Phi(\mathcal{U})$ the collection of morphisms u in $\mathcal{K}$ such that $C(u) \in \mathrm{Ob}(\mathcal{U})$.

Lemma 7.2. *Let* $\mathcal{U}$ *be an épaisse subcategory of* $\mathcal{K}$. *Then for* $f \in \mathcal{K}(X, Y)$ *the following are equivalent*.

(1) f *factors through* (*an object of*) $\mathcal{U}$.

(2) *There exists* $s \in \Phi(\mathcal{U})$ *such that* $sf = 0$.

(3) *There exists* $t \in \Phi(\mathcal{U})$ *such that* $ft = 0$.

Proof. (1) ⇒ (2). Let $f = vu$ for $u : X \to Z$, $v : Z \to Y$ with $Z \in \mathrm{Ob}(\mathcal{U})$. We have a triangle $(Y, C(v), TZ, s, \cdot, -Tv)$. Since $TZ \in \mathrm{Ob}(\mathcal{U})$, $s \in \Phi(\mathcal{U})$. Thus, since by Lemma 6.4 $T(s)T(v) = 0$, we have $sf = svu = 0$.

(2) $\Rightarrow$ (1). Let $sf = 0$ for $s \in \Phi(\mathcal{U})$. Then we have a triangle $(Y, Z, C(s), s, \cdot, w)$ and by Proposition 6.5 there exists $g : X \to T^{-1}C(s)$ such that $f = T^{-1}(w)g$.

(1) $\Leftrightarrow$ (3). Dual of (1) $\Leftrightarrow$ (2).

Definition 7.3. We call a square in $\mathcal{K}$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

a ∂-square if there exists a triangle of the form

$$\left(X, Y \oplus X', Y', \begin{bmatrix} u \\ f \end{bmatrix}, [g \quad -u'], \cdot\right).$$

Lemma 7.3. *The following hold*.

(1) ∂-*squares are commutative*.

(2) *Every diagram in* $\mathcal{K}$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \\ X' & & \end{array}$$

can be completed to a ∂-*square*.

(3) *Every diagram in* $\mathcal{K}$

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

can be completed to a ∂-*square*.

Proof. Obvious.

Lemma 7.4. *Let*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ f\downarrow & & \downarrow g \\ X' & \xrightarrow{u'} & Y' \end{array}$$

be a ∂-square with

$$(X, Y \oplus X', Y', \begin{bmatrix} u \\ f \end{bmatrix}, [g \quad -u'], \omega)$$

a triangle and embed u, u' *in triangles* (X, Y, Z, u, v, w) *and* (X', Y', Z', u', v', w'), *respectively. Then there exists an isomorphism* $h : Z \xrightarrow{\sim} Z'$ *such that* $\omega = wh^{-1}v'$ *and* (f, g, h) *is a homomorphism of triangles.*

Proof. Put $\hat{f} = {}^t[u \quad f]$, $\hat{g} = [g \quad -u']$. Since $[1 \ \ 0]\,\hat{f} = u$, by (TR4) and Lemma 6.8 we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\hat{f}} & Y \oplus X' & \xrightarrow{\hat{g}} & Y' & \xrightarrow{\omega} & TX \\ \| & & \downarrow \pi & & \downarrow \phi & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \hat{f}\downarrow & & \| & & \downarrow \psi & & \downarrow T\hat{f} \\ Y \oplus X' & \xrightarrow{\pi} & Y & \xrightarrow{0} & TX' & \xrightarrow{\mu} & TY \oplus TX' \\ \hat{g}\downarrow & & \downarrow v & & \| & & \downarrow T\hat{g} \\ Y' & \xrightarrow{\phi} & Z & \xrightarrow{\psi} & TX' & \xrightarrow{-Tu'} & TY' \end{array}$$

with the rows being triangles, where $\pi = [1 \ \ 0]$ and $\mu = {}^t[0 \ \ 1]$. Thus we get a homomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ f\downarrow & & \downarrow g & & \| & & \downarrow Tf \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{\phi} & Z & \xrightarrow{\psi} & TX'. \end{array}$$

Also, by (TR3) there exists $h : Z \to Z'$ which makes the following diagram commute

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{\phi} & Z & \xrightarrow{\psi} & TX' \end{array}$$

$$\begin{array}{ccccccc} \| & & \| & & \downarrow h & & \| \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'. \end{array}$$

Hence (f, g, h) is a homomorphism of triangles. By Proposition 6.6 h is an isomorphism, so that $\omega = w\phi = wh^{-1}v'$.

Lemma 7.5. *Every commutative square*

$$\begin{array}{ccc} X_1 & \xrightarrow{u_1} & Y_1 \\ f \downarrow & & \downarrow g \\ X_2 & \xrightarrow{u_2} & Y_2 \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & C(u_1) & \xrightarrow{w_1} & TX_1 \\ f \downarrow & & \downarrow g & & \downarrow & & \downarrow Tf \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & C(u_2) & \xrightarrow{w_2} & TX_2 \\ a \downarrow & & \downarrow b & & \downarrow & & \downarrow Ta \\ C(f) & \to & C(g) & \to & Z & \to & TC(f) \\ i \downarrow & & \downarrow j & & \downarrow & & \downarrow Ti \\ TX_1 & \xrightarrow{Tu_1} & TY_1 & \xrightarrow{Tv_1} & TC(u_1) & \xrightarrow{-Tw_1} & T^2X_1 \end{array}$$

with the rows and the columns except the right end being triangles.

Proof. Put $\hat{f} = {}^t[u_1 \quad f]$, $\hat{g} = [g \ -u_1]$ and $\hat{g}' = [g' \ -u_1]$. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{u_1} & Y_1 \\ f \downarrow & & \downarrow g' \\ X_2 & \xrightarrow{u_2'} & Y_2' \end{array}$$

be a ∂-square with $(X_1, Y_1 \oplus X_2, Y_2', \hat{f}, \hat{g}', \omega)$ a triangle. Since $\hat{g}\hat{f} = 0$, by Proposition 6.5 there exists $h : Y_2' \to Y_2$ such that $\hat{g} = h\hat{g}'$. Embed f, g' in triangles $(X_1, X_2, C(f), f, a, i)$ and $(Y_1, Y_2', C(g'), g', b', j)$, respectively. Then, since $(X_1, Y_1 \oplus X_2, Y_2', \hat{f}, -\hat{g}', -\omega)$ is a triangle, it follows by Lemma 7.4 that there exists an isomorphism $\sigma : C(f) \xrightarrow{\sim} C(g')$ such that $-\omega = i\sigma^{-1}b'$ and (u_1, u_2', σ) is a homomorphism of triangles. Thus by (TR4) we have a commutative diagram

$$
\begin{array}{ccccccccccc}
X_1 & \xrightarrow{u_1} & Y_1 & = & Y_1 & \xrightarrow{g'} & Y_2' & \xrightarrow{b'} & C(g') & \xrightarrow{\sigma^{-1}} & C(f) \\
f \downarrow & & \downarrow g' & & \downarrow g & & \downarrow h & & \downarrow \alpha & & \downarrow \alpha\sigma \\
X_2 & \xrightarrow{u_2'} & Y_2' & \xrightarrow{h} & Y_2 & = & Y_2 & \xrightarrow{b} & C(g) & = & C(g) \\
a \downarrow & & \downarrow b' & & \downarrow b & & \downarrow c & & \downarrow \beta & & \downarrow \beta \\
C(f) & \xrightarrow{\sigma} & C(g') & \xrightarrow{\alpha} & C(g) & \xrightarrow{\beta} & C(h) & = & C(h) & = & C(h) \\
i \downarrow & & \downarrow j' & & \downarrow j & & \downarrow k & & \downarrow T(b')k & & \downarrow T(\sigma^{-1}b')k \\
TX_1 & \xrightarrow{Tu_1} & TY_1 & = & TY_1 & \xrightarrow{Tg'} & TY_2' & \xrightarrow{Tb'} & TC(g') & \xrightarrow{T\sigma^{-1}} & TC(f)
\end{array}
$$

with the columns being triangles. Next, embed f, g' in triangles $(X_1, Y_1, C(u_1), u_1, v_1, w_1)$ and $(X_1, Y_2', C(u_2'), u_1', v_1', w_1')$, respectively. Then by Lemma 7.4 there exists an isomorphism $\tau : C(u_1) \xrightarrow{\sim} C(u_2')$ such that $\omega = w_1\tau^{-1}v_2'$ and (f, g', τ) is a homomorphism of triangles. Thus by (TR4) we have a commutative diagram

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & C(u_1) & \xrightarrow{w_1} & TX_1 \\
f \downarrow & & \downarrow g' & & \downarrow \tau & & \downarrow Tf \\
X_2 & \xrightarrow{u_2'} & Y_2' & \xrightarrow{v_2'} & C(u_2') & \xrightarrow{w_2'} & TX_2 \\
\| & & \downarrow g & & \downarrow \gamma & & \| \\
X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & C(u_2) & \xrightarrow{w_2} & TX_2 \\
u_2' \downarrow & & \| & & \downarrow \delta & & \downarrow Tu_2' \\
Y_2' & \xrightarrow{h} & Y_2 & \xrightarrow{c} & C(h) & \xrightarrow{k} & TY_2' \\
v_2' \downarrow & & \downarrow v_2 & & \| & & \downarrow Tv_2' \\
C(u_2') & \xrightarrow{\gamma} & C(u_2) & \xrightarrow{\delta} & C(h) & \xrightarrow{T(v_2')k} & TC(u_2') \\
\tau^{-1} \downarrow & & \| & & \| & & \downarrow T\tau^{-1} \\
C(u_1) & \xrightarrow{\gamma\tau} & C(u_2) & \xrightarrow{\delta} & C(h) & \xrightarrow{T(\tau^{-1}v_2')k} & TC(u_1)
\end{array}
$$

with the rows being triangles. Thus, since $-i\sigma^{-1}b' = \omega = w_1\tau^{-1}v_2'$, we get a commutative diagram

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & C(u_1) & \xrightarrow{w_1} & TX_1 \\
f \downarrow & & \downarrow g & & \downarrow \gamma\tau & & \downarrow Tf
\end{array}
$$

$$\begin{array}{ccccccc}
X_2 & \stackrel{u_2}{\to} & Y_2 & \stackrel{v_2}{\to} & C(u_2) & \stackrel{w_2}{\to} & TX_2 \\
a \downarrow & & \downarrow b & & \downarrow \delta & & \downarrow Ta \\
C(f) & \stackrel{\alpha\sigma}{\to} & C(g) & \stackrel{\beta}{\to} & C(h) & \xrightarrow{T(\tau^{-1}b')k} & TC(f) \\
i \downarrow & & \downarrow j & & \downarrow T(\sigma^{-1}v_2')k & & \downarrow Ti \\
TX_1 & \stackrel{Tu_1}{\to} & TY_1 & \stackrel{Tv_1}{\to} & TC(u_1) & \stackrel{-Tw_1}{\to} & T^2X_1
\end{array}$$

with the rows and the columns except the right end being triangles.

Definition 7.4. A multiplicative system in a category $\mathscr{C}$ is a collection S of morphisms in $\mathscr{C}$ which satisfies the following axioms:

(FR1) (1) $\mathrm{id}_X \in S$ for every $X \in \mathrm{Ob}(\mathscr{C})$.
(2) For $s, t \in S$, if st is defined, then $st \in S$.

(FR2) (1) Every diagram in $\mathscr{C}$

$$\begin{array}{ccc}
X & \stackrel{s}{\to} & Y \\
f \downarrow & & \\
X' & &
\end{array} \qquad \text{with } s \in S$$

can be completed to a commutative square

$$\begin{array}{ccc}
X & \stackrel{s}{\to} & Y \\
f \downarrow & & \downarrow g \\
X' & \stackrel{t}{\to} & Y'
\end{array} \qquad \text{with } s, t \in S.$$

(2) Every diagram in $\mathscr{C}$

$$\begin{array}{ccc}
 & & Y \\
 & & \downarrow g \\
X' & \stackrel{t}{\to} & Y'
\end{array} \qquad \text{with } t \in S$$

can be completed to a commutative square

$$X \stackrel{s}{\to} Y$$

$$\begin{array}{ccc} f\downarrow & & \downarrow g \\ X' & \xrightarrow{t} & Y' \end{array} \quad \text{with } s, t \in S.$$

(FR3) For $f, g \in \mathscr{C}(X, Y)$ the following are equivalent.

(1) There exists $s \in S$ such that $sf = sg$.

(2) There exists $t \in S$ such that $ft = gt$.

Definition 7.5. A multiplicative system S in a category $\mathscr{C}$ is called saturated if it satisfies the following axiom:

(FR0) For a morphism s in $\mathscr{C}$, if there exist f, g such that $sf, gs \in S$, then $s \in S$.

Definition 7.6. A multiplicative system S in a triangulated category $\mathscr{K}$ is said to be compatible with the triangulation if it satisfies the following axioms:

(FR4) For a morphism u in $\mathscr{K}$, $u \in S$ if and only if $Tu \in S$.

(FR5) For triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and morphisms $f : X \to X'$, $g : Y \to Y'$ in S with $gu = u'f$, there exists $h : Z \to Z'$ in S such that (f, g, h) is a homomorphism of triangles.

Proposition 7.6. *Let $\mathscr{U}$ be an épaisse subcategory of $\mathscr{K}$. Then $\Phi(\mathscr{U})$ is a saturated multiplicative system in $\mathscr{K}$ compatible with the triangulation.*

Proof. (FR0) (1) Let $f : X' \to X$, $s : X \to Y$, $g : Y \to Y'$ with $sf, gs \in \Phi(\mathscr{U})$. By (TR3) we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{s} & Y & \to & C(s) & \xrightarrow{u} & TX \\ \| & & \downarrow g & & \downarrow & & \| \\ X & \xrightarrow{gs} & Y' & \to & C(gs) & \to & TX. \end{array}$$

Also, by (TR4) we have a commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{f} & X & \xrightarrow{v} & C(f) & \to & TX' \\ \| & & \downarrow s & & \downarrow & & \| \\ X' & \xrightarrow{sf} & Y & \to & C(sf) & \to & TX' \end{array}$$

$$
\begin{array}{ccccccc}
f\downarrow & & \| & & \downarrow & & \downarrow Tf \\
X & \xrightarrow{s} & Y & \rightarrow & C(s) & \xrightarrow{u} & TX \\
v\downarrow & & \downarrow & & \| & & \downarrow Tv \\
C(f) & \rightarrow & C(sf) & \rightarrow & C(s) & \xrightarrow{w} & TC(f)
\end{array}
$$

with the rows being triangles. Note that $w = T(v)u$ factors through $C(gs) \in \mathrm{Ob}(\mathcal{U})$, and that $C(w) \cong TC(sf) \in \mathrm{Ob}(\mathcal{U})$. Thus $C(s) \in \mathrm{Ob}(\mathcal{U})$.

(FR1) (1) $C(\mathrm{id}_X) = 0 \in \mathrm{Ob}(\mathcal{U})$ for all $X \in \mathrm{Ob}(\mathcal{K})$.

(2) Let $t : X \to Y$, $s : Y \to Z$ be in $\Phi(\mathcal{U})$. By (TR4) we have a triangle of the form $(C(t), C(st), C(s), \cdot, \cdot, w)$. Since $C(s), TC(t) \in \mathrm{Ob}(\mathcal{U})$, $TC(st) \cong C(w) \in \mathrm{Ob}(\mathcal{U})$.

(FR2) By Lemmas 7.3 and 7.4.

(FR3) By Lemma 7.2.

(FR4) $C(T^n(s)) \cong T^nC(s) \in \mathrm{Ob}(\mathcal{U})$ for all $s \in \Phi(\mathcal{U})$ and $n \in \mathbb{Z}$.

(FR5) By Lemma 7.5.

Proposition 7.7. *Let $\mathcal{A}$ be an abelian category and $H : \mathcal{K} \to \mathcal{A}$ a cohomological functor. Let $\mathcal{U}$ be the full subcategory of $\mathcal{K}$ consisting of $X \in \mathrm{Ob}(\mathcal{K})$ with $H^n(X) = 0$ for all $n \in \mathbb{Z}$. Then the following hold.*

(1) *$\mathcal{U}$ is an épaisse subcategory of $\mathcal{K}$.*

(2) *For a morphism s in $\mathcal{K}$, $s \in \Phi(\mathcal{U})$ if and only if $H^n(s)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Proof. (1) It is obvious that $\mathcal{U}$ is stable under T. Let $u \in \mathcal{K}(X, Y)$. Then for each $n \in \mathbb{Z}$, we have an exact sequence

$$H^n(X) \to H^n(Y) \to H^n(C(u)) \to H^{n+1}(X) \to H^{n+1}(Y).$$

Thus, if $X, Y \in \mathrm{Ob}(\mathcal{U})$, then $C(u) \in \mathrm{Ob}(\mathcal{U})$. Thus $\mathcal{U}$ is a full triangulated subcategory of $\mathcal{K}$. Assume next that u factors through an object of $\mathcal{U}$ and that $C(u) \in \mathrm{Ob}(\mathcal{U})$. Then, since $H^n(u) = 0$ for all $n \in \mathbb{Z}$, it follows that $X, Y \in \mathrm{Ob}(\mathcal{U})$.

(2) “If” part. For any $n \in \mathbb{Z}$, since we have an exact sequence

$$H^n(X) \xrightarrow{\sim} H^n(Y) \to H^n(C(u)) \to H^{n+1}(X) \xrightarrow{\sim} H^{n+1}(Y),$$

we have $H^n(C(u))$.

“Only if” part. For any $n \in \mathbb{Z}$, since we have an exact sequence

$$H^{n-1}(C(u)) \to H^n(X) \to H^n(Y) \to H^n(C(u)),$$

$H^n(s)$ is an isomorphism.

Corollary 7.8. *Each* $X \in \operatorname{Ob}(\mathcal{K})$ *defines épaisse subcategories of* $\mathcal{K}$

$$\bigcap_{n \in \mathbb{Z}} \operatorname{Ker}\,(\mathcal{K}(X, -) \circ T^{n}), \qquad \bigcap_{n \in \mathbb{Z}} \operatorname{Ker}\,(\mathcal{K}(-, X) \circ T^{-n}).$$

Proof. By Proposition 6.5 $\mathcal{K}(X, -)$, $\mathcal{K}(-, X)$ are cohomological functors. Thus Proposition 7.7 applies.

Corollary 7.9. *Let* $X \in \operatorname{Ob}(\mathcal{K})$ *be a nonzero object and assume* $\mathcal{K}$ *has no proper épaisse subcategory* $\mathcal{U}$ *such that* $X \in \operatorname{Ob}(\mathcal{U})$. *Then the following hold.*

(1) *For any nonzero object* $Y \in \operatorname{Ob}(\mathcal{K})$ *there exists* $n \in \mathbb{Z}$ *such that* $\mathcal{K}(X, T^{n}Y) \quad 0$.

(2) *For any nonzero object* $Y \in \operatorname{Ob}(\mathcal{K})$ *there exists* $n \in \mathbb{Z}$ *such that* $\mathcal{K}(T^{-n}Y, X) \quad 0$.

Proof. (1) Suppose to the contrary that $\mathcal{K}(X, T^{n}Y) = 0$ for all $n \in \mathbb{Z}$. Then, since $\mathcal{K}(Y, Y)$ 0, and since $\mathcal{K}(T^{-n}X, Y) = 0$ for all $n \in \mathbb{Z}$, by Corollary 7.8 we have a proper épaisse subcategory

$$\mathcal{U} = \bigcap_{n \in \mathbb{Z}} \operatorname{Ker}\,(\mathcal{K}(-, Y) \circ T^{-n})$$

such that $X \in \operatorname{Ob}(\mathcal{U})$, a contradiction.

(2) Similar to (1).

Definition 7.7. Let $\mathcal{H}$ be another triangulated category. A ∂-functor $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ is a pair of an additive functor $F : \mathcal{K} \to \mathcal{H}$ and an isomorphism of functors $\theta : FT \xrightarrow{\sim} TF$ such that, for any triangle (X, Y, Z, u, v, w) in $\mathcal{K}$, $(FX, FY, FZ, Fu, Fv, \theta_X \circ Fw)$ is a triangle in $\mathcal{H}$.

A contravariant ∂-functor $F : \mathcal{K} \to \mathcal{H}$ is defined as a covariant ∂-functor $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{H}$.

Proposition 7.10. (1) *The identity functor* $\mathbf{1}_{\mathcal{K}} = (\mathbf{1}_{\mathcal{K}}, \mathrm{id})$ *is a* ∂*-functor*.

(2) *The translations* $T = (T, -\,\mathrm{id}_{T^2})$, $T^{-1} = (T^{-1}, -\,\mathrm{id}_{\mathbf{1}_{\mathcal{K}}}) : \mathcal{K} \to \mathcal{K}$ *are* ∂*-functors.*

(3) *Let* $F, G : \mathcal{K} \to \mathcal{H}$ *be functors and* $\sigma : F \to G$ *an isomorphism. Then, if* $F = (F, \theta)$ *is a* ∂*-functor, so is* $G = (G, T\sigma \circ \theta \circ \sigma_T^{-1})$. *Conversely, if* $G = (G, \eta)$ *is a* ∂*-functor, so is* $F = (F, T\sigma^{-1} \circ \eta \circ \sigma_T)$.

(4) *For any two consecutive* ∂*-functors* $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$, $G = (G, \eta) : \mathcal{H} \to \mathcal{G}$, *the composite* $GF = (GF, \eta_F \circ G\theta) : \mathcal{K} \to \mathcal{G}$ *is a* ∂*-functor*.

(5) *If* $F = (F, \theta)$ *is a* ∂*-functor, then* $T^nF = (T^nF, (-1)^n T^n\theta)$ *and* $FT^n = (FT^n, (-1)^n \theta_{T^n})$ *are* ∂*-functors for all* $n \in \mathbb{Z}$.

(6) *Let* $\mathcal{A}$, $\mathcal{B}$ *be abelian categories and* $F : \mathcal{A} \to \mathcal{B}$ *an additive functor. Then the extended functor* $F : K(\mathcal{A}) \to K(\mathcal{B})$ *is a* ∂*-functor.*

Proof. Straightforward.

Definition 7.8. Let $F = (F, \theta)$, $G = (G, \eta) : \mathcal{K} \to \mathcal{H}$ be ∂-functors. A homomorphism of ∂-functors $\zeta : (F, \theta) \to (G, \eta)$ is a homomorphism of functors $\zeta : F \to G$ such that $\eta \circ \zeta_T = T\zeta \circ \theta$. We denote by Hom (F, G) the collection of homomorphisms of ∂-functors $\zeta : (F, \theta) \to (G, \eta)$.

Proposition 7.11. (1) *If* $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ *is a* ∂*-functor, then* $\mathrm{id}_F \in$ Hom (F, F) *and* $\theta \in$ Hom (FT, TF).

(2) *If* $F, G : \mathcal{K} \to \mathcal{H}$ *are* ∂*-functors, then* $\xi - \zeta \in$ Hom (F, G) *for all* $\xi, \zeta \in$ Hom (F, G).

(3) *If* $F, G, H : \mathcal{K} \to \mathcal{H}$ *are* ∂*-functors, then* $\xi \circ \zeta \in$ Hom (F, H) *for all* $\zeta \in$ Hom (F, G) *and* $\xi \in$ Hom (G, H).

(4) *If* $F, G : \mathcal{K} \to \mathcal{H}$ *and* $H : \mathcal{H} \to \mathcal{G}$ *are* ∂*-functors, then* $H\zeta \in$ Hom (HF, HG) *for all* $\zeta \in$ Hom (F, G).

(5) *If* $H : \mathcal{G} \to \mathcal{K}$ *and* $F, G : \mathcal{K} \to \mathcal{H}$ *are* ∂*-functors, then* $\zeta_H \in$ Hom (FH, GH) *for all* $\zeta \in$ Hom (F, G).

Proof. Straightforward.

Proposition 7.12. *Let* $F : \mathcal{K} \to \mathcal{H}$ *be a* ∂*-functor and* $\mathcal{U} = \mathrm{Ker}\, F$ *the full subcategory of* $\mathcal{K}$ *consisting of* $X \in \mathrm{Ob}(\mathcal{K})$ *with* $FX = 0$. *Then the following hold.*

(1) $\mathcal{U}$ *is an épaisse subcategory of* $\mathcal{K}$.

(2) *For a morphism* s *in* $\mathcal{K}$, $s \in \Phi(\mathcal{U})$ *if and only if* $F(s)$ *is an isomorphism.*

Proof. (1) It is obvious that $\mathcal{U}$ is stable under T. Let $(X, Y, Z, u, \cdot, \cdot)$ be a triangle in $\mathcal{K}$. Since we have a triangle in $\mathcal{H}$ of the form $(FX, FY, FZ, F(u), \cdot, \cdot)$, if $FX = FY = 0$ then by Lemma 6.9 $FZ = 0$. Thus $\mathcal{U}$ is a full triangulated subcategory of $\mathcal{K}$. Assume next that u factors through an object of $\mathcal{U}$ and that $FZ = 0$. Then, since $F(u) = 0$, and since by Lemma 6.9 $F(u)$ is an isomorphism, it follows that $FX = FY = 0$.

(2) By Lemma 6.9.

Proposition 7.13. *Let* $\mathcal{A}$ *be an abelian category and* $\mathcal{L}$ *a subcollection of* $\mathrm{Ob}(\mathcal{A})$ *containing zero objects and closed under finite direct sums. Let* $* = +, -, \mathrm{b}$ *or nothing and let* $F = (F, \theta) : C^*(\mathcal{L}) \to \mathcal{K}$ *be an additive functor together with an isomorphism of functors* $\theta : FT \xrightarrow{\sim} TF$. *Assume* $(FX^\bullet, FY^\bullet, FC(u), Fu, Fv, \theta_X \circ Fw)$ *is a triangle in* $\mathcal{K}$ *for all mapping cylinder* $(X^\bullet, Y^\bullet, C(u), u, v, w)$ *in* $C^*(\mathcal{L})$. *Then* F *factors through* $K^*(\mathcal{L})$ *and the induced functor* $K^*(\mathcal{L}) \to \mathcal{K}$ *is a* ∂*-functor.*

Proof. Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(C^*(\mathcal{L}))$ and $u \in \mathrm{Htp}(X^\bullet, Y^\bullet)$. We have only to show $Fu = 0$.

Let ($X^\bullet$, $Y^\bullet$, $C(u)$, u, v, w) the mapping cylinder of u, where $v = {}^t[0 \quad 1]$ and $w = [1 \quad 0]$. Then by Proposition 3.1 v is a section, so is Fv. Thus, it follows by Lemma 6.8 that $Fu = 0$.

§8. Quotient categories

Throughout this section, $\mathscr{C}$ is a category and S is a multiplicative system in $\mathscr{C}$. Unless otherwise stated, functors are covariant functors.

Definition 8.1. For a morphism $f : X \to Y$, we set source(f) = X and sink(f) = Y.

Definition 8.2. For each $X \in \mathrm{Ob}(\mathscr{C})$, we have a category S^X such that

$$\mathrm{Ob}(S^X) = \{s \in S \mid \mathrm{source}(s) = X\},$$

$$S^X(s, s') = \{f \in \mathscr{C}(\mathrm{sink}(s), \mathrm{sink}(s')) \mid s' = fs\} \text{ for } s, s' \in \mathrm{Ob}(S^X),$$

and a category S_X such that

$$\mathrm{Ob}(S_X) = \{t \in S \mid \mathrm{sink}(t) = X\},$$

$$S_X(t, t') = \{f \in \mathscr{C}(\mathrm{source}(t), \mathrm{source}(t')) \mid t = t'f\} \text{ for } t, t' \in \mathrm{Ob}(S_X).$$

Lemma 8.1. *For any* $X \in \mathrm{Ob}(\mathscr{C})$, S^X *satisfies the following axioms*:

(L1) For any $f_1 \in S^X(s, s_1')$, $f_2 \in S^X(s, s_2')$, there exist $s'' \in S^X$ and $g_1 \in S^X(s_1', s'')$, $g_2 \in S^X(s_2', s'')$ such that $g_1f_1 = g_2f_2$.

(L2) For any $f_1, f_2 \in S^X(s, s')$, there exist $s'' \in S^X$ and $g \in S^X(s', s'')$ such that $gf_1 = gf_2$.

(L3') S^X has an initial object id_X.

Proof. (L1) Let $s : X \to Y$, $s_1' : X \to Y_1'$, $s_2' : X \to Y_2'$. Then $s_i' = f_is$ for $i = 1, 2$. By (FR2) there exist $h_i : Y_i' \to Z$ such that $h_1s_1' = h_2s_2'$ and $h_1 \in S$. Since $(h_1f_1)s = h_1s_1' = h_2s_2' = (h_2f_2)s$ with $s \in S$, by (FR3) there exists $t : Z \to Y''$ in S such that $(th_1)f_1 = t(h_1f_1) = t(h_2f_2) = (th_2)f_2$. Put $s'' = th_1s_1' = th_2s_2'$. Then by (FR1) $s'' \in \mathrm{Ob}(S^X)$ and $th_i \in S^X(s_i', s'')$ for $i = 1, 2$.

(L2) Let $s : X \to Y$, $s' : X \to Y'$. Since $f_1s = s' = f_2s$, by (FR3) there exists $g : Y' \to Z$ in S such that $gf_1 = gf_2$. Put $s'' = gs'$. Then by (FR1) $s'' \in \mathrm{Ob}(S^X)$ and $g \in S^X(s', s'')$.

(L3') By (FR1) $\mathrm{id}_X \in \mathrm{Ob}(S^X)$.

Corollary 8.2. *Let* $X \in \mathrm{Ob}(\mathscr{C})$ *and* $s_1, s_2, \cdots, s_n \in \mathrm{Ob}(S^X)$. *Then there exists* $s' \in \mathrm{Ob}(S^X)$ *such that* $S^X(s_i, s')$ Ø *for all* $1 \le i \le n$.

Proof. By (L1) and (L3').

Definition 8.3. For X, $Y \in \mathrm{Ob}(\mathscr{C})$, we have a covariant functor

$$\mathscr{C}(X, -) : S^Y \to (Sets),$$

where $\mathscr{C}(X, s) = \mathscr{C}(X, \mathrm{sink}(s))$ for $s \in \mathrm{Ob}(S^Y)$, and a contravariant functor

$$\mathscr{C}(-, Y) : S_X \to (Sets),$$

where $\mathscr{C}(t, Y) = \mathscr{C}(\mathrm{source}(t), Y)$ for $t \in \mathrm{Ob}(S_X)$.

Lemma 8.3. *Let* X, $Y \in \mathrm{Ob}(\mathscr{C})$. *Define a relation* $\sim$ *on the collection*

$$\{(f, s) \mid s \in \mathrm{Ob}(S^Y), f \in \mathscr{C}(X, \mathrm{sink}(s))\}$$

as follows: $(f_1, s_1) \sim (f_2, s_2)$ *if and only if there exist* $h_1 \in S^Y(s_1, s')$, $h_2 \in S^Y(s_2, s')$ *such that* $(h_1f_1, s') = (h_2f_2, s')$. *Then* $\sim$ *is an equivalence relation and we have*

$$\varinjlim_{S^Y} \mathscr{C}(X, -) = \{(f, s) \mid s \in \mathrm{Ob}(S^Y), f \in \mathscr{C}(X, \mathrm{sink}(s))\}/\sim.$$

Proof. It only remains to check the trnsitivity. Let $(f_1, s_1) \sim (f_2, s_2)$, $(f_2, s_2) \sim (f_3, s_3)$. Then, there exist $g_1 \in S^X(s_1, s_1')$, $g_2 \in S^X(s_2, s_1')$ such that $g_1f_1 = g_2f_2$, and there exist $g_2' \in S^X(s_2, s'')$, $g_3 \in S^X(s_3, s'')$ such that $g_2'f_2 = g_3f_3$. Thus, since by (L1) there exist $h_1 \in S^X(s_1', s'')$, $h_2 \in S^X(s_2', s'')$ such that $h_1g_2 = h_2g_2'$, we have $h_1g_1f_1 = h_1g_2f_2 = h_2g_2'f_2 = h_2g_3f_3$, so that $(f_1, s_1) \sim (f_3, s_3)$.

Definition 8.4. For X, $Y \in \mathrm{Ob}(\mathscr{C})$, we denote by $[(f, s)]$ the equivalence class of (f, s) with $s \in \mathrm{Ob}(S^Y)$, $f \in \mathscr{C}(X, \mathrm{sink}(s))$.

Lemma 8.4. *For any* X, Y, $Z \in \mathrm{Ob}(S^{-1}\mathscr{C})$ *we have a well-defined mapping*

$$\varinjlim_{S^Y} \mathscr{C}(X, -) \times \varinjlim_{S^Z} \mathscr{C}(Y, -) \to \varinjlim_{S^Z} \mathscr{C}(X, -)$$

which is defined as follows: *with each pair* $([(f, s)], [(g, t)])$, *since by* (FR2) *there exist* $s' \in S$ *with* $\mathrm{source}(s') = \mathrm{sink}(t)$, $g' \in \mathscr{C}(\mathrm{sink}(s), \mathrm{sink}(s'))$ *such that* $g's = s'g$, *we associate the equivalence class* $[(g'f, s't)]$.

Proof. Straightforward.

Definition 8.5. We define a category $S^{-1}\mathscr{C}$, called the quotient category of $\mathscr{C}$, as follows: (1) $\mathrm{Ob}(S^{-1}\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$; (2) for $X, Y \in \mathrm{Ob}(\mathscr{C})$, the morphism set is given by

$$S^{-1}\mathscr{C}(X, Y) = \varinjlim_{S^Y} \mathscr{C}(X, -);$$

(3) for $X, Y, Z \in \mathrm{Ob}(S^{-1}\mathscr{C})$, the law of composition is given by

$$S^{-1}\mathscr{C}(X, Y) \times S^{-1}\mathscr{C}(Y, Z) \to S^{-1}\mathscr{C}(X, Z),\ ([(f, s)], [(g, t)]) \mapsto [(g'f, s't)],$$

where $[(g', s')] \in S^{-1}\mathscr{C}(\mathrm{sink}(s), \mathrm{sink}(t))$ with $g's = s'g$; and (4) the identity of $X \in \mathrm{Ob}(S^{-1}\mathscr{C})$ is given by the equivalence class $[(\mathrm{id}_X, \mathrm{id}_X)]$.

Definition 8.6. We have a functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$, called the canonical functor, such that

$$Q(X) = X \text{ for } X \in \mathrm{Ob}(\mathscr{C}),$$
$$Q(f) = [(f, \mathrm{id}_Y)] \text{ for } f \in \mathscr{C}(X, Y).$$

Lemma 8.5. *$Q : \mathscr{C} \to S^{-1}\mathscr{C}$ takes terminal objects to terminal objects.*

Proof. Let $Y \in \mathrm{Ob}(\mathscr{C})$ be a terminal object. Denote by ξ_X the unique element of $\mathscr{C}(X, Y)$ for $X \in \mathrm{Ob}(\mathscr{C})$. Then $[(f, s)] = [(\xi_Z f, \xi_Z s)] = [(\xi_X, \mathrm{id}_Y)]$ for all (f, s) with $s \in \mathscr{C}(Y, Z) \cap S$ and $f \in \mathscr{C}(X, Z)$.

Proposition 8.6. *For $f, g \in \mathscr{C}(X, Y)$ the following are equivalent.*

(1) $Q(f) = Q(g)$.

(2) *There exists $s \in \mathrm{Ob}(S^Y)$ such that $sf = sg$.*

(3) *There exists $t \in \mathrm{Ob}(S_X)$ such that $ft = gt$.*

Proof. (1) $\Rightarrow$ (2). Note that $S^Y(\mathrm{id}_Y, s) = \{s\}$ for all $s \in \mathrm{Ob}(S^Y)$. Thus by definition there exists $s \in \mathrm{Ob}(S^Y)$ such that $(sf, s) = (sg, s)$.

(2) $\Rightarrow$ (1). We have $Q(f) = [(f, \mathrm{id}_Y)] = [(sf, s)] = [(sg, s)] = [(g, \mathrm{id}_Y)] = Q(g)$.

(2) $\Leftrightarrow$ (3). By (FR3).

Proposition 8.7. *The following hold.*

(1) *$Q(s)$ is an isomorphism for all $s \in S$.*

(2) *For any $X, Y \in \mathrm{Ob}(S^{-1}\mathscr{C})$ we have*

$$S^{-1}\mathscr{C}(X, Y) = \{Q(s)^{-1}Q(f) \mid s \in \mathrm{Ob}(S^Y), f \in \mathscr{C}(X, \mathrm{sink}(s))\}$$
$$= \{Q(g)Q(t)^{-1} \mid t \in \mathrm{Ob}(S_X), g \in \mathscr{C}(\mathrm{source}(t), Y)\}.$$

Proof. (1) Let $s \in S$ with $\mathrm{source}(s) = X$, $\mathrm{sink}(s) = Y$. Then by definition $Q(s) \circ [(\mathrm{id}_Y, s)] = [(\mathrm{id}_Y, \mathrm{id}_Y)]$. Also, $[(\mathrm{id}_Y, s)] \circ Q(s) = [(s, s)] = [(\mathrm{id}_X, \mathrm{id}_X)]$.

(2) For any (f, s) with $s \in \mathrm{Ob}(S^Y)$ and $f \in \mathscr{C}(X, \mathrm{sink}(s))$, since $Q(s) \circ [(f, s)] = Q(f)$, by the part (1) we get $[(f, s)] = Q(s)^{-1}Q(f)$. Also, since by (FR2) there exist $t \in \mathrm{Ob}(S_X)$, $g \in \mathscr{C}(\mathrm{source}(t), Y)$ such that $ft = sg$, $Q(f)Q(t) = Q(s)Q(g)$ and by the part (1) we get $Q(g)Q(t)^{-1} = Q(s)^{-1}Q(f)$.

Proposition 8.8. *For $f \in \mathscr{C}(X, Y)$ the following are equivalent.*

(1) *$Q(f)$ is an isomorphism.*

(2) *There exist morphisms g, h in $\mathscr{C}$ with $gf, fh \in S$.*

Proof. (1) $\Rightarrow$ (2). There exist $s \in S$ with $\mathrm{source}(s) = X$, $g \in \mathscr{C}(Y, \mathrm{sink}(s))$ such that $Q(f)^{-1} = Q(s)^{-1}Q(g)$. Since $Q(s) = Q(g)Q(f) = Q(gf)$, $[(s, \mathrm{id}_X)] = [(gf, \mathrm{id}_X)]$ and there exists $s' \in \mathrm{Ob}(S^X)$ such that $s's = s'gf$. Then by (FR2) $(s'g)f \in S$. Dually, there exist $t \in S$ with $\mathrm{sink}(t) = Y$, $h \in \mathscr{C}(\mathrm{source}(t), X)$ and $t' \in \mathrm{Ob}(S_Y)$ such that $f(ht') \in S$.

(2) $\Rightarrow$ (1). Since $Q(g)Q(f) = Q(gf)$ has a left inverse, so does $Q(f)$. Also, since $Q(f)Q(h) = Q(fh)$ has a right inverse, so does $Q(f)$.

Corollary 8.9. *Assume S is saturated. Then for any $f \in \mathscr{C}(X, Y)$ the following hold.*

(1) *$Q(f)$ is an isomorphism if and only if $f \in S$.*

(2) *If there exists $s \in \mathrm{Ob}(S^Y)$ with $sf \in S$, then $f \in S$.*

(3) *If there exists $t \in \mathrm{Ob}(S_X)$ with $ft \in S$, then $f \in S$.*

Proof. (1) By Propositions 8.7(1) and 8.8.

(2) By Proposition 8.8 $Q(f) = Q(s)^{-1}Q(sf)$ is an isomorphism, thus by the part (1) $f \in S$.

(3) By Proposition 8.8 $Q(f) = Q(ft)Q(t)^{-1}$ is an isomorphism, thus by the part (1) $f \in S$.

Proposition 8.10. *Let $\mathscr{D}$ be another category and $F : \mathscr{C} \to \mathscr{D}$ a functor such that $F(s)$ is an isomorphism for all $s \in S$. Then there exists a unique functor $F' : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $F = QF'$.*

Proof. By Proposition 8.7(2).

Proposition 8.11. *Let $\mathscr{D}$ be another category and $F, G : S^{-1}\mathscr{C} \to \mathscr{D}$ functors. Then we have a bijective correspondence*

$$\mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(FQ, GQ),\ \tau \mapsto \tau_Q,$$

where Hom(–, –) *denotes the collection of homomorphisms of functors*.

Proof. Since $\mathrm{Ob}(S^{-1}\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$, we may consider Hom($F$, G) as a subcollection of Hom(FQ, GQ). Let $\sigma \in \mathrm{Hom}(FQ, GQ)$. For any $\phi = Q(s)^{-1}Q(f) \in S^{-1}\mathscr{C}(X, Y)$ with sink(s) = Z, since we have a commutative diagram in $\mathscr{D}$

$$\begin{array}{ccccc} FQX & \xrightarrow{FQf} & FQZ & \xleftarrow{FQs} & FQY \\ \sigma_X \downarrow & & \downarrow \sigma_Z & & \downarrow \sigma_Y \\ GQX & \xrightarrow{GQf} & GQZ & \xleftarrow{GQs} & GQY, \end{array}$$

$\sigma_Y \circ F\phi = G\phi \circ \sigma_X$. Thus $\sigma \in \mathrm{Hom}(F, G)$.

Lemma 8.12 (Dual of Lemma 8.1). *For any* $X \in \mathrm{Ob}(\mathscr{C})$, S_X *satisfies the following axioms*:

(L1)$^{\circ}$ For any $g_1 \in S_X(s_1', s'')$, $g_2 \in S_X(s_2', s'')$, there exist $s \in S_X$ and $f_1 \in S_X(s, s_1')$, $f_2 \in S_X(s, s_2')$ such that $g_1 f_1 = g_2 f_2$.

(L2)$^{\circ}$ For any $g_1, g_2 \in S_X(s', s'')$, there exist $s \in S_X$ and $f \in S_X(s, s')$ such that $g_1 f = g_2 f$.

(L3')$^{\circ}$ S_X has a terminal object id_X.

Corollary 8.13 (Dual of Corollary 8.2). *Let* $X \in \mathrm{Ob}(\mathscr{C})$ *and* $t_1, t_2, \cdots, t_n \in \mathrm{Ob}(S_X)$. *Then there exists* $t' \in \mathrm{Ob}(S_X)$ *such that* $S_X(t', t_i)$ Ø *for all* $1 \le i \le n$.

Lemma 8.14 (Dual of Lemma 8.3). *Let* $X, Y \in \mathrm{Ob}(\mathscr{C})$. *Define a relation* $\sim$ *on the collection*

$$\{(t, g) \mid t \in \mathrm{Ob}(S_X), g \in \mathscr{C}(\mathrm{source}(t), Y)\}$$

as follows: $(t_1, g_1) \sim (t_2, g_2)$ *if and only if there exist* $h_1 \in S_X(t', t_1)$, $h_2 \in S_X(t', t_2)$ *such that* $(t', g_1 h_1) = (t', g_2 h_2)$. *Then* $\sim$ *is an equivalence relation and we have*

$$\varinjlim_{S_X} \mathscr{C}(-, Y) = \{(t, g) \mid t \in \mathrm{Ob}(S_X), g \in \mathscr{C}(\mathrm{source}(t), Y)\}/\sim.$$

Definition 8.7. For X, $Y \in \mathrm{Ob}(\mathscr{C})$, we denote also by $[(t, g)]$ the equivalence class of (t, g) with $t \in \mathrm{Ob}(S_X)$, $g \in \mathscr{C}(\mathrm{source}(t), Y)$.

Proposition 8.15. *For any* X, $Y \in \mathrm{Ob}(\mathscr{C})$ *we have a bijection*

$$\theta = \theta_{X,Y} : \varinjlim_{S^Y} \mathscr{C}(X, -) \xrightarrow{\sim} \varinjlim_{S_X} \mathscr{C}(-, Y)$$

which associates with each $[(f, s)]$ *the equivalence class of* (t, g) *such that* $ft = sg$.

Proof. Let (f_1, s_1), (f_2, s_2) with $[(f_1, s_1)] = [(f_2, s_2)]$, and let (t_1, g_1), (t_2, g_2) with $f_1t_1 = s_1g_1$, $f_2t_2 = s_2g_2$. We claim $[(t_1, g_1)] = [(t_2, g_2)]$. By definition, there exist $h_1 \in S^Y(s_1, s')$, $h_2 \in S^Y(s_2, s')$ such that $(h_1f_1, s') = (h_2f_2, s')$. Put $h' = h_1f_1 = h_2f_2$. Then by (FR2) there exist $t' \in \mathrm{Ob}(S_X)$ and $g' \in \mathscr{C}(\mathrm{source}(t'), Y)$ such that $h't' = s'g'$. Again by (FR2), there exist $j \in S$ with $\mathrm{sink}(j) = \mathrm{source}(t')$ and $j_1 \in \mathscr{C}(\mathrm{source}(j), \mathrm{source}(t_1))$ such that $t_1j_1 = t'j$. Since $s'g'j = h't'j = h_1f_1t'j = h_1f_1t_1j_1 = h_1s_1g_1j_1 = s'g_1j_1$, by (FR3) there exists $j' \in S$ such that $g'jj' = g_1j_1j'$. Note also that by (FR1) $t_1j_1j' = t'jj' \in \mathrm{Ob}(S_X)$. Thus $[(t_1, g_1)] = [(t', g')]$. Similarly, $[(t_2, g_2)] = [(t', g')]$, so that $[(t_1, g_1)] = [(t_2, g_2)]$. Thus $\theta_{X,Y}$ is well-defined. Dually, we have a well-defined mapping

$$\eta = \eta_{X,Y} : \varinjlim_{S_X} \mathscr{C}(-, Y) \to \varinjlim_{S^Y} \mathscr{C}(X, -)$$

which associates with each $[(t, g)]$ the equivalence class of (f, s) such that $ft = sg$. It is obvious that $\eta_{X,Y}$ is the inverse of $\theta_{X,Y}$.

Remark 8.1. For $[(f, s)] \in \varinjlim_{S^Y} \mathscr{C}(X, -)$ and $[(t, g)] \in \varinjlim_{S_X} \mathscr{C}(-, Y)$ the following are equivalent.

(1) $\theta_{X,Y}([(f, s)]) = [(t, g)]$.

(2) $Q(s)^{-1}Q(f) = Q(g)Q(t)^{-1}$.

Remark 8.2. Let X, Y, $Z \in \mathrm{Ob}(\mathscr{C})$. Define a law of composition

$$\varinjlim_{S_X} \mathscr{C}(-, Y) \times \varinjlim_{S_Y} \mathscr{C}(-, Z) \to \varinjlim_{S_X} \mathscr{C}(-, Z)$$

as follows: with each pair $([(s, f)], [(t, g)])$, since by (FR2) there exist $t' \in S$ with $\mathrm{sink}(t') = \mathrm{source}(s)$ and $f' \in \mathscr{C}(\mathrm{source}(t'), \mathrm{source}(t))$ such that $tf' = ft'$, we associate the equivalence class $[(st', gf')]$. Then the isomorphism in Proposition 8.15 is compatible with the law of composition.

Lemma 8.16 (Dual of Lemma 8.5). *$Q : \mathscr{C} \to S^{-1}\mathscr{C}$ takes initial objects to initial objects.*

Proposition 8.17. *Let $\mathscr{D}$ be a full subcategory of $\mathscr{C}$. Assume $S \cap \mathscr{D}$ is a multiplicative system in $\mathscr{D}$ and one of the following conditions is satisfied*:

(1) *For any $s \in$ Ob(S^Y) with $Y \in$ Ob$(\mathscr{D})$, there exists $f \in \mathscr{C}$(sink(s), Y') with $Y' \in$ Ob$(\mathscr{D})$ such that $fs \in S$.*

(2) *For any $t \in$ Ob(S_X) with $X \in$ Ob$(\mathscr{D})$, there exists $g \in \mathscr{C}(X'$, source$(t))$ with $X' \in$ Ob$(\mathscr{D})$ such that $tg \in S$.*

Then the canonical functor $(S \cap \mathscr{D})^{-1}\mathscr{D} \to S^{-1}\mathscr{C}$ is fully faithful, so that $(S \cap \mathscr{D})^{-1}\mathscr{D}$ can be considered as a fullsubcategory of $S^{-1}\mathscr{C}$.

Proof. Straightforward.

Proposition 8.18. *Assume $\mathscr{C}$ is an additive category. Then $S^{-1}\mathscr{C}$ is an additive category and $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ is an addtive functor.*

Proof. We divide the proof into several steps.

Claim 1: $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ takes zero objects to zero objects.

Proof. By Lemmas 8.5 and 8.16.

Let X, $Y \in$ Ob$(S^{-1}\mathscr{C})$. We now define an addition on $S^{-1}\mathscr{C}(X, Y)$. For each pair of morphisms $[(f_1, s_1)]$, $[(f_2, s_2)] \in S^{-1}\mathscr{C}(X, Y)$, since by Corollary 8.2 there exist $s' \in$ Ob(S^Y) and $g_1 \in S^Y(s_1, s')$, $g_2 \in S^Y(s_2, s')$, we can define the sum of them as follows

$$[(f_1, s_1)] + [(f_2, s_2)] = [(g_1f_1 + g_2f_2, s')].$$

Claim 2: The addition above is well-defined.

Proof. Let $[(f_1, s_1)] = [(f_1', s_1')]$ and $[(f_2, s_2)] = [(f_2', s_2')]$. According to Corollary 8.2, we may assume $s_1 = s_2 = s$ and $s_1' = s_2' = s'$. We claim $[(f_1 + f_2, s)] = [(f_1' + f_2', s')]$. By (L1) and (L3') there exist $t_i \in$ Ob(S^Y) and $g_i \in S^Y(s, t_i)$, $g_i' \in S^Y(s', t_i)$ such that $g_if_i = g_i'f_i'$ for $i = 1, 2$, then by (L1) there exist $t' \in$ Ob(S^Y) and $h_1 \in S^Y(t_1, t')$, $h_2 \in S^Y(t_2, t')$ such that $h_1g_1 = h_2g_2$, and then by (L2) there exist $t'' \in$ Ob(S^Y) and $j \in S^Y(t', t'')$ such that $jh_1g_1' = jh_2g_2'$. Thus we get

$$\begin{aligned}[(f_1 + f_2, s)] &= [(jh_1g_1(f_1 + f_2), t'')] \\ &= [(jh_1g_1f_1 + jh_1g_1f_2, t'')] \\ &= [(jh_1g_1f_1 + jh_2g_2f_2, t'')]\end{aligned}$$

$= [(jh_1g_1'f_1' + jh_2g_2'f_2', t'')]$
$= [(jh_1g_1'f_1' + jh_1g_1'f_2', t'')]$
$= [(jh_1g_1'(f_1' + f_2'), t'')]$
$= [(f_1' + f_2', s')]$.

Claim 3: $S^{-1}\mathscr{C}(X, Y)$ is an additive group with $0 = [(0, \mathrm{id}_Y)]$ and $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ induces a homomorphism of additive groups $\mathscr{C}(X, Y) \to S^{-1}\mathscr{C}(X, Y)$.

Proof. By definition, $[(f, \mathrm{id}_Y)] + [(g, \mathrm{id}_Y)] = [(f + g, \mathrm{id}_Y)]$ for all $f, g \in \mathscr{C}(X, Y)$. Next, for any $[(f, s)] \in S^{-1}\mathscr{C}(X, Y)$, since $[(0, \mathrm{id}_Y)] = [(0, s)]$, we have

$$[(f, s)] + [(0, \mathrm{id}_Y)] = [(f, s)] + [(0, s)] = [(f, s)],$$

so that $[(0, \mathrm{id}_Y)]$ is the zero element of $S^{-1}\mathscr{C}(X, Y)$. Also, for any $[(f, s)] \in S^{-1}\mathscr{C}(X, Y)$, since

$$[(f, s)] + [(-f, s)] = [(0, s)] = [(0, \mathrm{id}_Y)],$$

we have $-[(f, s)] = [(-f, s)]$.

Claim 4: The law of composition is bilinear.

Proof. Straightforward.

Remark 8.3. Assume $\mathscr{C}$ is an additive category. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$. Define an addition on $\varinjlim_{S_X} \mathscr{C}(-, Y)$ as follows: for each pair of $[(t_1, g_1)], [(t_2, g_2)] \in \varinjlim_{S_X} \mathscr{C}(-, Y)$, since by Corollary 8.13 there exist $t' \in \mathrm{Ob}(S_X)$ and $f_1 \in S_X(t', t_1)$, $f_2 \in S_X(t', t_2)$, we set

$$[(t_1, g_1)] + [(t_2, g_2)] = [(t', g_1f_1 + g_2f_2)].$$

Then the isomorphism in Proposition 8.15 is compatible with the addtion.

Proposition 8.19. *Assume $\mathscr{C}$ is an additive category. Then for $f \in \mathscr{C}(X, Y)$ the following are equivalent.*

(1) $Q(f) = 0$.

(2) *There exists $s \in \mathrm{Ob}(S^Y)$ such that* $sf = 0$.

(3) *There exists $t \in \mathrm{Ob}(S_X)$ such that* $ft = 0$.

Proof. By Proposition 8.6.

Corollary 8.20. *Assume $\mathscr{C}$ is an additive category. Then for $X \in \mathrm{Ob}(\mathscr{C})$ the following are equivalent.*

(1) $Q(X) = 0$.

(2) $\mathrm{id}_{Q(X)} = Q(\mathrm{id}_X) = 0$.

(3) S^X *contains a zero morphism*.

(4) S_X *contains a zero morphism*.

Proposition 8.21. *Assume $\mathscr{C}$ is an additive category. Let $\mathscr{D}$ be another additive category and $F : \mathscr{C} \to \mathscr{D}$ an additive functor such that $F(s)$ is an isomorphism for all $s \in S$. Then there exists a unique additive functor $F' : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $F = QF'$.*

Proof. By Proposition 8.10 there exists a unique functor $F' : S^{-1}\mathscr{C} \to \mathscr{D}$ such that $F = QF'$. It follows by the definition of addition in $S^{-1}\mathscr{C}$ that F' is additive.

§9. Quotient categories of triangulated categories

Throughout this section, $\mathcal{K}$ is a triangulated category, $\mathcal{U}$ is an épaisse subcategory of $\mathcal{K}$ and $S = \Phi(\mathcal{U})$ is the collection of morphisms u in $\mathcal{K}$ with $C(u) \in \mathcal{U}$. Unless otherwise stated, functors are covariant functors.

Lemma 9.1. *S is a saturated multiplicative system in $\mathcal{K}$ compatible with the triangulation.*

Proof. By Proposition 7.6.

Definition 9.1. We denote by $\mathcal{K}/\mathcal{U}$ the quotient category $S^{-1}\mathcal{K}$ and by $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ the canonical functor.

Lemma 9.2. *$\mathcal{K}/\mathcal{U}$ is an additive category and $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ is an additive functor.*

Proof. By Proposition 8.18.

Proposition 9.3. (1) *For a morphism u in $\mathcal{K}$, $Q(u)$ is an isomorphism if and only if $u \in S$.*
(2) *For a morphism u in $\mathcal{K}$, $Q(u) = 0$ if and only if u factors through (an object of) $\mathcal{U}$.*
(3) *$\mathcal{U} = \mathrm{Ker}\, Q$, i.e., $\mathcal{U}$ consists of the objects $X \in \mathrm{Ob}(\mathcal{K})$ with $QX = 0$.*

Proof. (1) By Proposition 7.6 and Corollary 8.9(1).

"Only if" part of (2). Let $X = \mathrm{source}(u)$. By Proposition 8.19 there exist $t \in \mathrm{Ob}(S_X)$ such that $ut = 0$. Then by Proposition 6.5 u factors through $C(t) \in \mathrm{Ob}(\mathcal{U})$.

(3) Let $X \in \mathrm{Ob}(\mathcal{U})$ and $0_X : 0 \to X$ the zero morphism. Since $C(0_X) \cong X \in \mathrm{Ob}(\mathcal{U})$, $0_X \in \mathrm{Ob}(S_X)$. Thus by Proposition 8.7(1) $Q(0_X)$ is an isomorphism and $0 = QX$. Conversely, let $X \in \mathrm{Ob}(\mathcal{K})$ with $QX = 0$. Since $Q(\mathrm{id}_X) = 0$, by the "only if" part of (1) id_X factors through (an object of) $\mathcal{U}$. Also, by (TR1) $C(\mathrm{id}_X) = 0 \in \mathrm{Ob}(\mathcal{U})$. Thus $X \in \mathrm{Ob}(\mathcal{U})$.

"If" part of (2). Assume u factors through $Z \in \mathrm{Ob}(\mathcal{U})$. Then by the part (2) $Q(u)$ factors through $QZ = 0$, so that $Q(u) = 0$.

Proposition 9.4. *For any $u \in \mathcal{K}/\mathcal{U}(X, Y)$ the following are equivalent.*
(1) *u is an isomorphism.*
(2) *$u = Q(s)^{-1}Q(f)$ with $s, f \in S$.*
(3) *$u = Q(g)Q(t)^{-1}$ with $g, t \in S$.*

Proof. By Lemma 9.1 and Corollary 8.9.

Lemma 9.5. *The trnslation $T : \mathcal{K} \to \mathcal{K}$ induces an autofunctor $\mathcal{K}/\mathcal{U} \to \mathcal{K}/\mathcal{U}$, which we* *denote also by T.*

Proof. By (FR4) and Proposition 8.10.

Remark 9.1. The canonical functor $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ commutes with the translation T, so that Q takes cylinders into cylinders.

Lemma 9.6. *Let* $(X, Y, Z, u, \cdot, \cdot)$ *and* $(X', Y', Z', u', \cdot, \cdot)$ *be triangles in* $\mathcal{K}$ *and let* $f \in \mathcal{K}(X, X')$, $g \in \mathcal{K}(Y, Y')$ *with* $Q(g)Q(u) = Q(u')Q(f)$. *Then there exists* $\phi \in \mathcal{K}/\mathcal{U}(Z, Z')$ *which makes the following diagram in* $\mathcal{K}/\mathcal{U}$ *commute*

$$\begin{array}{ccccccc} QX & \xrightarrow{Q(u)} & QY & \longrightarrow & QZ & \longrightarrow & TQX \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(g)} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle TQ(f)} \\ QX' & \xrightarrow{Q(u')} & QY' & \longrightarrow & QZ' & \longrightarrow & TQX'. \end{array}$$

Proof. Since $Q(u'f - gu) = 0$, by Proposition 9.3(2) $u'f - gu$ factors through some $W \in \mathrm{Ob}(\mathcal{U})$. Let $v \in \mathcal{K}(X, W)$, $w \in \mathcal{K}(W, Y')$ with $u'f - gu = wv$. Set $\hat{u} = {}^t[u \;\; v] : X \to Y \oplus W$, $\hat{g} = [g \;\; w] : Y \oplus W \to Y'$ and $s = [1 \;\; 0] : Y \oplus W \to Y$. Then $s\hat{u} = u$ and $\hat{g}\hat{u} = u'f$. Thus by (TR3) we get homomorphisms of triangles in $\mathcal{K}$

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \to & Z & \to & TX \\ \| & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \| \\ X & \xrightarrow{\hat{u}} & Y \oplus W & \to & Z'' & \to & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{g}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ X' & \xrightarrow{u'} & Y' & \to & Z' & \to & TX'. \end{array}$$

Furthermore, by (FR5) we may assume $t \in S$. Thus, since by Proposition 9.3(3) $Q(s) = \mathrm{id}_{QY}$ and $Q(\hat{g}) = Q(g)$, it follows that $\phi = Q(h)Q(t)^{-1}$ is a desired morphism.

Lemma 9.7. *Let*

$$\begin{array}{ccc} QX & \xrightarrow{Q(u)} & QY \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ QX' & \xrightarrow{Q(u')} & QY' \end{array}$$

be a commutative diagram in $\mathcal{K}/\mathcal{U}$. *Let* $\phi = Q(s)^{-1}Q(f)$ *and* $X'' = \mathrm{sink}(s)$. *Then there exist* u''

$\in \mathcal{K}(X'', Y'')$, $t \in \mathcal{K}(Y', Y'') \cap S$ *and* $g \in \mathcal{K}(Y, Y'')$ *such that*

(1) $u''s = tu'$,

(2) $\psi = Q(t)^{-1}Q(g)$, *and*

(3) $Q(g)Q(u) = Q(u'')Q(f)$.

Proof. Let $\psi = Q(t')^{-1}Q(g')$ with $Z = \text{sink}(t')$. By (FR2) there exist $v \in \mathcal{K}(X'', Z')$ and $s' \in \mathcal{K}(Y', Z') \cap S$ such that $vs = s'u'$. Then again by (FR2) there exist $s'' \in \mathcal{K}(Z', Y'')$ and $t'' \in \mathcal{K}(Z, Y'') \cap S$ such that $t''t' = s''s'$. Put $t = t''t'$, $g = t''g'$ and $u'' = s''v$. Then by (FR1) $t \in S$ and we have

(1)
$$\begin{aligned} tu' &= t''t'u' \\ &= s''s'u' \\ &= s''vs \\ &= u''s, \end{aligned}$$

(2)
$$\begin{aligned} Q(t)^{-1}Q(g) &= Q(t''t')^{-1}Q(t''g') \\ &= Q(t')^{-1}Q(t'')^{-1}Q(t'')Q(g') \\ &= Q(t')^{-1}Q(g') \\ &= \psi \end{aligned}$$

and

(3)
$$\begin{aligned} Q(g)Q(u) &= Q(t'')Q(g')Q(u) \\ &= Q(t'')Q(t')\psi Q(u) \\ &= Q(t'')Q(t')\, Q(u')\phi \\ &= Q(t'')Q(t')Q(u')Q(s)^{-1}Q(f) \\ &= Q(t'')Q(t')Q(s')^{-1}Q(v)Q(f) \\ &= Q(t'')Q(t'')^{-1}Q(s'')Q(v)Q(f) \\ &= Q(u'')Q(f). \end{aligned}$$

Definition 9.2. A cylinder $(QX', QY', QZ', \lambda, \mu, \nu)$ in $\mathcal{K}/\mathcal{U}$ is called a triangle if there exists a triangle (X, Y, Z, u, v, w) in $\mathcal{K}$ such that $(QX', QY', QZ', \lambda, \mu, \nu)$ is isomorphic to (QX, QY, QZ, Qu, Qv, Qw).

Proposition 9.8. *$\mathcal{K}/\mathcal{U}$ is a triangulated category and $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ is a ∂-functor.*

Proof. It remains to check that $\mathcal{K}/\mathcal{U}$ satisfies the axioms (TR1)-(TR4).

(TR1) Let $\lambda \in \mathcal{K}/\mathcal{U}(X, Y)$. Let $\lambda = Q(s)^{-1}Q(u)$ with $\text{sink}(u) = Y'$ and embed u in a triangle

$(X, Y', Z, u, \cdot, \cdot)$ in $\mathcal{K}$. Then we have an isomorphism of cylinders

$$\begin{array}{ccccccc}
QX & \xrightarrow{\lambda} & QY & \longrightarrow & QZ & \longrightarrow & TQX \\
\| & & \downarrow Q(s) & & \| & & \| \\
QX & \xrightarrow{Q(u)} & QY' & \longrightarrow & QZ & \longrightarrow & TQX .
\end{array}$$

Also, for any $X \in \mathrm{Ob}(\mathcal{K}/\mathcal{U})$, since $Q(\mathrm{id}_X) = \mathrm{id}_{QX}$, $(QX, QX, 0, \mathrm{id}_{QX}, 0, 0)$ is a triangle in $\mathcal{K}/\mathcal{U}$.

(TR2) By the fact that $QT = TQ$.

(TR3) Let $(X, Y, Z, u, \cdot, \cdot)$, $(X', Y', Z', u', \cdot, \cdot)$ be triangles in $\mathcal{K}$ and $\alpha \in \mathcal{K}/\mathcal{U}(X, X')$, $\beta \in \mathcal{K}/\mathcal{U}(Y, Y')$ with $\beta Q(u) = Q(u')\alpha$. Let $\alpha = Q(s)^{-1}Q(f)$ with $\mathrm{sink}(s) = X''$. Then by Lemma 9.7 there exist $u'' \in \mathcal{K}(X'', Y'')$, $t \in \mathcal{K}(Y', Y'') \cap S$ and $g \in \mathcal{K}(Y, Y'')$ such that $u''s = tu'$, $\beta = Q(t)^{-1}Q(g)$ and $Q(g)Q(u) = Q(u'')Q(f)$. Let $(X'', Y'', Z'', u'', \cdot, \cdot)$ be a triangle in $\mathcal{K}$. Then by Lemma 9.6 we have a commutative diagram

$$\begin{array}{ccccccc}
QX & \xrightarrow{Q(u)} & QY & \longrightarrow & QZ & \longrightarrow & TQX \\
Q(f)\downarrow & & \downarrow Q(g) & & \downarrow \phi & & \downarrow TQ(f) \\
QX'' & \xrightarrow{Q(u'')} & QY'' & \longrightarrow & QZ'' & \longrightarrow & TQX'' .
\end{array}$$

Also, by (FR5) we have a commutative diagram

$$\begin{array}{ccccccc}
X'' & \xrightarrow{u''} & Y'' & \rightarrow & Z'' & \rightarrow & TX'' \\
s\uparrow & & \uparrow t & & \uparrow q & & \uparrow Ts \\
X' & \xrightarrow{u'} & Y' & \rightarrow & Z' & \rightarrow & TX' .
\end{array}$$

Thus, setting $\gamma = Q(q)^{-1}\phi$, we get a commutative diagram

$$\begin{array}{ccccccc}
QX & \xrightarrow{Q(u)} & QY & \longrightarrow & QZ & \longrightarrow & TQX \\
\alpha\downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow T\alpha \\
QX' & \xrightarrow{Q(u'')} & QY' & \longrightarrow & QZ' & \longrightarrow & TQX' .
\end{array}$$

(TR4) Let $\phi : QX \to QY$, $\psi : QY \to QZ$ be consecutive morphisms in $\mathcal{K}/\mathcal{U}$. Let $\phi = Q(u)Q(s)^{-1}$ with $\mathrm{source}(s) = X'$ and $\psi = Q(t)^{-1}Q(v)$ with $\mathrm{sink}(t) = Z'$. Then we have a commutative diagram

$$\begin{array}{ccccc} QX & \xrightarrow{\phi} & QY & \xrightarrow{\psi} & QZ \\ Q(s)^{-1}\downarrow & & \| & & \downarrow Q(t) \\ QX' & \xrightarrow{Q(u)} & QY & \xrightarrow{Q(v)} & QZ', \end{array}$$

so that we may assume $\phi = Q(u)$ and $\psi = Q(v)$. Thus, since $\mathcal{K}$ satisfies (TR4), so does $\mathcal{K}/\mathcal{U}$.

Proposition 9.9. *Let $\mathcal{A}$ be an abelian category and $H : \mathcal{K} \to \mathcal{A}$ a cohomological functor vanishing on $\mathcal{U}$, i.e., $HX = 0$ for all $X \in \mathrm{Ob}(\mathcal{U})$. Then there exists a unique cohomological functor $\overline{H} : \mathcal{K}/\mathcal{U} \to \mathcal{A}$ such that $H = \overline{H}Q$.*

Proof. Let $s \in S$. We claim that $H(s)$ is an isomorphism. Put $X = \mathrm{source}(s)$ and $Y = \mathrm{sink}(s)$. Since $C(s) \in \mathrm{Ob}(\mathcal{U})$, $H(T^n(C(s))) = 0$ for all $n \in \mathbb{Z}$. Thus, since we have an exact sequence

$$H(T^{-1}(C(s))) \longrightarrow HX \xrightarrow{H(s)} HY \longrightarrow H(T(C(s))),$$

$H(s)$ is an isomorphism. Hence by Proposition 8.21 there exists a unique additive functor $\overline{H} : \mathcal{K}/\mathcal{U} \to \mathcal{A}$ such that $H = \overline{H}Q$. It is obvious that $\overline{H} : \mathcal{K}/\mathcal{U} \to \mathcal{A}$ is a cohomological functor.

Proposition 9.10. *Let $\mathcal{H}$ be another triangulated category and $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ a ∂-functor vanishing on $\mathcal{U}$, i.e., $FX = 0$ for all $X \in \mathrm{Ob}(\mathcal{U})$. Then there exists a unique ∂-functor $\overline{F} = (\overline{F}, \overline{\theta}) : \mathcal{K}/\mathcal{U} \to \mathcal{H}$ such that $F = \overline{F}Q$ and $\theta = \overline{\theta}_Q$.*

Proof. Let $s \in S$. We claim that $F(s)$ is an isomorphism. Put $X = \mathrm{source}(s)$ and $Y = \mathrm{sink}(s)$. Since $C(s) \in \mathrm{Ob}(\mathcal{U})$, $F(C(s)) = 0$. Thus $(FX, FY, 0, F(s), 0, 0)$ is a triangle and by Lemma 6.9 $F(s)$ is an isomorphism. Hence by Proposition 8.21 there exists a unique additive functor $\overline{F} : \mathcal{K}/\mathcal{U} \to \mathcal{H}$ such that $F = \overline{F}Q$. Since $QT = TQ$, we have an isomorphism $\theta : \overline{F}QT = \overline{F}TQ \xrightarrow{\sim} T\overline{F}Q$. Thus by Proposition 8.11 we have an isomorphism $\overline{\theta} : \overline{F}T \xrightarrow{\sim} T\overline{F}$ such that $\theta = \overline{\theta}_Q$. It is obvious that $\overline{F} = (\overline{F}, \overline{\theta}) : \mathcal{K}/\mathcal{U} \to \mathcal{H}$ is a ∂-functor.

Proposition 9.11. *Let $\mathcal{H}$ be another triangulated category and $F = (F, \theta)$, $G = (G, \eta) : \mathcal{K}/\mathcal{U} \to \mathcal{H}$ ∂-functors. Then we have a bijective correspondence*

$$\mathrm{Hom}\,(F, G) \xrightarrow{\sim} \mathrm{Hom}\,(FQ, GQ),\ \zeta \mapsto \zeta_Q.$$

Proof. For any $\zeta \in \mathrm{Hom}\,(F, G)$, since

$$\eta_Q \circ (\zeta_Q)_T = \eta_Q \circ \zeta_{QT}$$

$$
\begin{aligned}
&= \eta_Q \circ \zeta_{TQ} \\
&= (\eta \circ \zeta_T)_Q \\
&= (T\zeta \circ \theta)_Q \\
&= T(\zeta_Q) \circ \theta_Q,
\end{aligned}
$$

we have $\zeta_Q \in$ Hom (FQ, GQ). Conversely, let $\xi \in$ Hom (FQ, GQ). Then by Proposition 8.11 there exists a unique $\zeta \in$ Hom(F, G) such that $\xi = \zeta_Q$. We claim $\zeta \in$ Hom (F, G). Since

$$
\begin{aligned}
(\eta \circ \zeta_T)_Q &= \eta_Q \circ \zeta_{TQ} \\
&= \eta_Q \circ \zeta_{QT} \\
&= \eta_Q \circ \xi_T \\
&= T\xi \circ \theta_Q \\
&= T(\zeta_Q) \circ \theta_Q \\
&= (T\zeta \circ \theta)_Q,
\end{aligned}
$$

by Proposition 8.11 $\eta \circ \zeta_T = T\zeta \circ \theta$ and $\zeta \in$ Hom (F, G).

Definition 9.3. An object $Y \in$ Ob$(\mathcal{K})$ is called $\mathcal{U}$-local if $\mathcal{K}(-, Y)$ *vanishes on* $\mathcal{U}$.

Proposition 9.12. *The collection of $\mathcal{U}$-local objects forms an épaisse subcategory and is closed under direct products, i.e., for a family of $\mathcal{U}$-local objects $\{Y_\lambda\}_{\lambda \in \Lambda}$, if the direct product $\prod Y_\lambda$ exists in $\mathcal{K}$, then $\prod Y_\lambda$ is $\mathcal{U}$-local.*

Proof. By Corollary 7.8 the first assertion follows. Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be a family of $\mathcal{U}$-local objects such that $\prod Y_\lambda$ exists in $\mathcal{K}$. Then $\mathcal{K}(X, \prod Y_\lambda) \cong \prod \mathcal{K}(X, Y_\lambda) = 0$ for all $X \in$ Ob$(\mathcal{U})$ and $\prod Y_\lambda$ is $\mathcal{U}$-local.

Proposition 9.13. *For any $\mathcal{U}$-local object $Y \in$ Ob$(\mathcal{K})$ the following hold.*

(1) *For any $t \in \mathcal{K}(X', X) \cap S$, $\mathcal{K}(t, Y)$ is an isomorphism. In particular, if $Y' \in$ Ob$(\mathcal{K})$ is another $\mathcal{U}$-local object, then every $s \in \mathcal{K}(Y, Y') \cap S$ is an isomorphism.*

(2) *The canonical functor $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ induces an isomorphism of functors on $\mathcal{K}$*

$$\mathcal{K}(-, Y) \xrightarrow{\sim} \mathcal{K}/\mathcal{U}(-, QY) \circ Q.$$

Proof. (1) Embed t in a triangle $(X', X, Z, t, \cdot, w)$ in $\mathcal{K}$. Since $Z, T^{-1}Z \in$ Ob$(\mathcal{U})$, it follows by Proposition 6.5(2) that $\mathcal{K}(t, Y)$ is an isomorphism. Next, let $s \in \mathcal{K}(Y, Y') \cap S$ with $Y' \in$ Ob$(\mathcal{K})$ $\mathcal{U}$-local. Then by the above there exists $s' \in \mathcal{K}(Y', Y)$ such that $s's = \mathrm{id}_Y$. It then follows by Lemma 9.1 that $s' \in S$. Thus, again by the above, there exists $s'' \in \mathcal{K}(Y, Y')$ such that $s''s' = \mathrm{id}_{Y'}$. Hence $s'' = s''s's = s$ and $s' = s^{-1}$.

(2) Let $X \in \mathrm{Ob}(\mathcal{K})$. Let $f \in \mathcal{K}(X, Y)$ with $Q(f) = 0$. Then by Proposition 9.3(2) f factors through some $Z \in \mathrm{Ob}(\mathcal{U})$ and $f = 0$. Conversely, let $u = Q(g) \circ Q(t)^{-1} \in \mathcal{K}/\mathcal{U}(QX, QY)$ with $t \in \mathcal{K}(X', X) \cap S$. Then by the part (1) there exists $f \in \mathcal{K}(X, Y)$ such that $g = ft$. It follows that $u = Q(f)$.

Proposition 9.14. *Assume $\mathcal{U}$ is closed under direct products, i.e., for a family of objects $\{Z_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{U}$, if the direct product $\prod Z_\lambda$ exists in $\mathcal{K}$, then $\prod Z_\lambda \in$ $\mathrm{Ob}(\mathcal{U})$. Then the canonical functor $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ preserves direct products. In particular, if $\mathcal{K}$ has arbitrary direct products, so does $\mathcal{K}/\mathcal{U}$.*

Proof. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of objects in $\mathcal{K}$ and assume the direct product $\prod X_\lambda$ exists in $\mathcal{K}$. For each $\mu \in \Lambda$ we denote by $p_\mu : \prod X_\lambda \to X_\mu$ the projection. We claim that for any $Y \in \mathrm{Ob}(\mathcal{K}/\mathcal{U})$ the canonical homomorphism

$$\xi_Y : \mathcal{K}/\mathcal{U}(Y, Q(\prod X_\lambda)) \to \prod \mathcal{K}/\mathcal{U}(Y, Q(X_\lambda)),\ u \mapsto (Q(p_\lambda) \circ u)$$

is an isomorphism.

Claim 1: ξ_Y is an epimorphism.

Proof. Let $(u_\lambda) \in \prod \mathcal{K}/\mathcal{U}(Y, Q(X_\lambda))$. For each $\mu \in \Lambda$, let $u_\mu = Q(s_\mu)^{-1} \circ Q(f_\mu)$ with $s_\mu \in \mathcal{K}(X_\mu, X_\mu') \cap S$ and embed s_μ in a triangle $(X_\mu, X_\mu', Z_\mu, s_\mu, \cdot, \cdot)$ in $\mathcal{K}$. Then by Proposition 6.10 the direct product

$$(\prod X_\lambda', \prod X_\lambda, \prod Z_\lambda, \prod s_\lambda, \cdot, \cdot)$$

is a triangle in $\mathcal{K}$. Also, since $Z_\mu \in \mathrm{Ob}(\mathcal{U})$ for all $\mu \in \Lambda$, $\prod Z_\lambda \in \mathrm{Ob}(\mathcal{U})$ and $\prod s_\lambda \in S$. For each $\mu \in \Lambda$, we denote by $p_\mu' : \prod X_\lambda' \to X_\mu'$ the projection. There exists $f \in \mathcal{K}(Y, \prod X_\lambda')$ such that $f_\mu = p_\mu' \circ f$ for all $\mu \in \Lambda$. Set $u = Q(\prod s_\lambda)^{-1} \circ Q(f)$. Then for any $\mu \in \Lambda$ we have

$$\begin{aligned} Q(p_\mu) \circ u &= Q(p_\mu) \circ Q(\prod s_\lambda)^{-1} \circ Q(f) \\ &= Q(s_\mu)^{-1} \circ Q(p_\mu') \circ Q(f) \\ &= Q(s_\mu)^{-1} \circ Q(f_\mu) \\ &= u_\mu. \end{aligned}$$

Claim 2: ξ_Y is a monomorphism.

Proof. Let $u \in \mathcal{K}/\mathcal{U}(Y, Q(\prod X_\lambda))$ with $Q(p_\mu) \circ u = 0$ for all $\mu \in \Lambda$. Let $u = Q(g) \circ Q(t)^{-1}$ with $t \in \mathcal{K}(Y', Y) \cap S$. We claim $Q(g) = 0$. For any $\lambda \in \Lambda$, since $Q(p_\lambda \circ g) \circ Q(t)^{-1} = 0$, we have $Q(p_\lambda \circ g) = 0$ and by Proposition 9.3(2) $p_\lambda \circ g$ factors through some $Z_\lambda \in \mathrm{Ob}(\mathcal{U})$. It follows that g factors through $\prod Z_\lambda \in \mathrm{Ob}(\mathcal{U})$. Thus again by Proposition 9.3(2) $Q(g) = 0$.

Remark 9.2. Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4* and $\mathcal{U}$ the épaisse subcategory of $K(\mathcal{A})$ consisting of acyclic complexes. Then $K(\mathcal{A})$ has arbitrary direct products and $\mathcal{U}$ is closed under direct products.

Definition 9.4. An object $X \in \mathrm{Ob}(\mathcal{K})$ is called $\mathcal{U}$-colocal if $\mathcal{K}(X, -)$ vanishes on $\mathcal{U}$.

Proposition 9.15 (Dual of Proposition 9.12). *The collection of $\mathcal{U}$-colocal objects forms an épaisse subcategory which is closed under direct sums, i.e., for a family of $\mathcal{U}$-colocal objects $\{X_\lambda\}_{\lambda \in \Lambda}$, if the direct sum $\oplus X_\lambda$ exists in $\mathcal{K}$, then $\oplus X_\lambda$ is $\mathcal{U}$-colocal*.

Proposition 9.16 (Dual of Proposition 9.13). *For any $\mathcal{U}$-colocal object $X \in \mathrm{Ob}(\mathcal{K})$ the following hold*.

(1) *For any $s \in \mathcal{K}(Y, Y') \cap S$, $\mathcal{K}(X, s)$ is an isomorphism. In particular, if $X' \in \mathrm{Ob}(\mathcal{K})$ is another $\mathcal{U}$-colocal object, then every $t \in \mathcal{K}(X', X) \cap S$ is an isomorphism.*

(2) *For any $\mathcal{U}$-colocal object $X \in \mathrm{Ob}(\mathcal{K})$ the canonical functor $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ induces an isomorphism of functors on $\mathcal{K}$*

$$\mathcal{K}(X, -) \xrightarrow{\sim} \mathcal{K}/\mathcal{U}(QX, -) \circ Q.$$

Proposition 9.17 (Dual of Proposition 9.14). *Assume $\mathcal{U}$ is closed under direct sums, i.e., for a family of objects $\{Z_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{U}$, if the direct sum $\oplus Z_\lambda$ exists in $\mathcal{K}$, then $\oplus Z_\lambda \in \mathrm{Ob}(\mathcal{U})$. Then the canonical functor $Q : \mathcal{K} \to \mathcal{K}/\mathcal{U}$ preserves direct sums. In particular, if $\mathcal{K}$ has arbitrary direct products, so does $\mathcal{K}/\mathcal{U}$.*

Remark 9.3. Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4 and $\mathcal{U}$ the épaisse subcategory of $K(\mathcal{A})$ consisting of acyclic complexes. Then $K(\mathcal{A})$ has arbitrary direct sums and $\mathcal{U}$ is closed under direct sums.

§10. Derived categories

Throughout this section, $\mathscr{A}$ is an abelian category, $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$ and $\mathscr{U}$ is the full subcategory of $K(\mathscr{A})$ consisting of acyclic complexes, i.e., $\mathscr{U} = \operatorname{Ker} H^\bullet$.

Lemma 10.1. *For* $* = +, -, \mathrm{b}, (+, \mathrm{b}), (-, \mathrm{b})$ *or nothing, the following hold.*
(1) $\mathscr{U} \cap K^*(\mathscr{A})$ *is an épaisse subcategory of* $K^*(\mathscr{A})$.
(2) $\Phi(\mathscr{U} \cap K^*(\mathscr{A}))$ *consists of the quasi-isomorphisms in* $K^*(\mathscr{A})$.
(3) $\Phi(\mathscr{U} \cap K^*(\mathscr{A})) = \Phi(\mathscr{U}) \cap K^*(\mathscr{A})$.

Proof. By Propositions 5.2 and 7.7.

Definition 10.1. For $* = +, -, \mathrm{b}, (+, \mathrm{b}), (-, \mathrm{b})$ or nothing, according to Lemma 10.1(1), we have a quotient category

$$D^*(\mathscr{A}) = K^*(\mathscr{A})/\mathscr{U} \cap K^*(\mathscr{A}),$$

called the derived category of $\mathscr{A}$. We denote by $Q : K^*(\mathscr{A}) \to D^*(\mathscr{A})$ the canonical functor.

Proposition 10.2. *For a morphism* $f \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ *the following are equivalent.*
(1) $Q(f) = 0$.
(2) *There exists a quasi-isomorphism* $s \in K(\mathscr{A})(Y^\bullet, Y'^\bullet)$ *such that* $sf = 0$.
(3) *There exists a quasi-isomorphism* $t \in K(\mathscr{A})(X'^\bullet, X^\bullet)$ *such that* $ft = 0$.
(4) f *factors through an acyclic complex*.

Proof. By Propositions 8.6 and 9.3(2).

Remark 10.1. For $X^\bullet \in \operatorname{Ob}(D(\mathscr{A}))$, since $\mathscr{U} = \operatorname{Ker} H^\bullet$, by Proposition 9.3(3) $H^\bullet(X^\bullet) = 0$ if and only if $X^\bullet = 0$. However, for $u \in D(\mathscr{A})(X^\bullet, Y^\bullet)$, $H^\bullet(u) = 0$ does not necessarily imply $u = 0$, i.e., for $f \in K(\mathscr{A})(X^\bullet, Y^\bullet)$, $H^\bullet(f) = 0$ does not necessarily imply $Q(f) = 0$. Consider for example the following homomorphism $f : X^\bullet \to Y^\bullet$ in $C(\operatorname{Mod} \mathbb{Z})$:

$$\begin{array}{ccccccccccc}
\cdots & \to & 0 & \to & \mathbb{Z} & \xrightarrow{d_X^0} & \mathbb{Z} & \to & 0 & \to & \cdots \\
 & & \downarrow & & f^0 \downarrow & & \downarrow f^1 & & \downarrow & & \\
\cdots & \to & 0 & \to & \mathbb{Z} & \xrightarrow{d_Y^0} & \mathbb{Z}/3\mathbb{Z} & \to & 0 & \to & \cdots ,
\end{array}$$

where $d_X^0(n) = 2n$, $d_Y^0(n) = n \bmod 3$, $f^0(n) = n$, and $f^1(n) = 2n \bmod 3$ for $n \in \mathbb{Z}$. Then we have $H^\bullet(f) = 0$. Let $t : X'^\bullet \to X^\bullet$ be a quasi-isomorphism. Let $x \in Z^1(X'^\bullet)$ such that $t^1(x) \notin B^1(X^\bullet) = 2\mathbb{Z}$. Since $t^1(2x) = 2t^1(x) \in B^1(X^\bullet)$, $2x \in B^1(X'^\bullet)$ and there exists $x' \in X'^0$ such that $2x = d^0_{X'}(x')$. Then $2t^0(x') = d^0_X(t^0(x')) = t^1(d^0_{X'}(x')) = t^1(2x) = 2t^1(x)$, so that $t^0(x') = t^1(x)$. Let $h^0 : X'^1 \to Y^0 = \mathbb{Z}$. If $t^0 = f^0 \circ t^0 = h^0 \circ d^0_{X'}$, then $t^1(x) = t^0(x') = h(d^0_{X'}(x')) = h(2x) = 2h^0(x) \in B^1(X^\bullet)$. Consequently, there can not exist $h : ft \simeq 0$.

Proposition 10.3. *For a morphism* $f \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ *the following are equivalent.*

(1) *$Q(f)$ is an isomorphism.*

(2) *f is a quasi-isomorphism.*

Proof. By Proposition 9.3(1).

Proposition 10.4. *For a morphism* $u \in D(\mathscr{A})(X^\bullet, Y^\bullet)$ *the following are equivalent.*

(1) *u is an isomorphism.*

(2) $u = Q(s)^{-1}Q(f)$ *with* $s, f \in \Phi(\mathscr{U})$.

(3) $u = Q(g)Q(t)^{-1}$ *with* $g, t \in \Phi(\mathscr{U})$.

(4) *$H^\bullet(u)$ is an isomorphism.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By Proposition 9.4.

(1) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). Embed u in a triangle $(X^\bullet, Y^\bullet, Z^\bullet, u, \cdot, \cdot)$. For any $n \in \mathbb{Z}$, since we have an exact sequence

$$H^n(X^\bullet) \xrightarrow{\sim} H^n(Y^\bullet) \to H^n(Z^\bullet) \to H^{n+1}(X^\bullet) \xrightarrow{\sim} H^{n+1}(Y^\bullet),$$

$H^n(Z^\bullet) = 0$. Thus $Z^\bullet$ is acyclic, so that $Z^\bullet = 0$ in $D(\mathscr{A})$. It follows by Lemma 6.9 that u is an isomorphism.

Definition 10.2. For each $n \in \mathbb{Z}$, we define truncation functors $\sigma_{>n}$, $\sigma_{\ n} : C(\mathscr{A}) \to C(\mathscr{A})$ as follows:

$$\sigma_{>n}(X^\bullet)^i = \begin{cases} X^i & (i > n) \\ B'^n(X^\bullet) & (i = n), \\ 0 & (i < n) \end{cases} \qquad \sigma_{\ n}(X^\bullet)^i = \begin{cases} 0 & (i > n) \\ Z^n(X^\bullet) & (i = n) \\ X^i & (i < n) \end{cases}$$

for $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. We set $\sigma_{\geq n} = \sigma_{>n-1}$ and $\sigma_{<n} = \sigma_{\ n-1}$.

Lemma 10.5. *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following hold.*

(1) *There exists a natural exact sequence* $0 \to \sigma_{\ n}(X^\bullet) \to X^\bullet \to \sigma_{>n}(X^\bullet) \to 0$.

(2) $$H^i(\sigma_{>n}(X^\bullet)) = \begin{cases} H^i(X^\bullet) & (i > n) \\ 0 & (i \le n) \end{cases}.$$

(3) $$H^i(\sigma_{\ n}(X^\bullet)) = \begin{cases} 0 & (i > n) \\ H^i(X^\bullet) & (i \le n) \end{cases}.$$

Proof. Straightforward.

Lemma 10.6. *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) *The canonical epimorphism* $X^\bullet \to \sigma_{>n}(X^\bullet)$ *is a quasi-isomorphism.*

(2) $\sigma_{\ n}(X^\bullet)$ *is acyclic.*

(3) $H^i(X^\bullet) = 0$ *for* $i \le n$.

Proof. (1) $\Leftrightarrow$ (2). By Lemma 10.5(1) and Proposition 4.3(1).

(2) $\Leftrightarrow$ (3). By Lemma 10.5(3).

Lemma 10.7 (Dual of Lemma 10.6). *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) *The canonical monomorphism* $\sigma_{\ n}(X^\bullet) \to X^\bullet$ *is a quasi-isomorphism.*

(2) $\sigma_{>n}(X^\bullet)$ *is acyclic.*

(3) $H^i(X^\bullet) = 0$ *for* $i > n$.

Lemma 10.8. *For any* $n \in \mathbb{Z}$ *we have well-defined truncation functors*

$$\sigma_{>n} : D(\mathscr{A}) \to D^+(\mathscr{A}) \quad \textit{and} \quad \sigma_{\ n} : D(\mathscr{A}) \to D^-(\mathscr{A}).$$

Proof. Let $n \in \mathbb{Z}$. For any $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ we denote by

$$\pi_X^n : X^n \to B''^n(X^\bullet) \quad \text{and} \quad \mu_X^n : B'^n(X^\bullet) \to X^{n+1}$$

the canonical epimorphism and the inclusion, respectively. Let $u \in \mathrm{Htp}(X^\bullet, Y^\bullet)$ and $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, Y^\bullet)$ with $h : u \simeq 0$. Define $h' \in \mathscr{A}^{\mathbb{Z}}(T(\sigma_{>n}(X^\bullet)), \sigma_{>n}(Y^\bullet))$ as follows:

$$h'^i = \begin{cases} h^i & (i > n) \\ \pi_Y^n \circ h^n \circ \mu_X^n & (i = n) \\ 0 & (i < n) \end{cases}.$$

Then it is easy to check that $h' : \sigma_{>n}(u) \simeq 0$. Thus we get a well-defined functor $\sigma_{>n} : K(\mathscr{A}) \to K^+(\mathscr{A})$. Next, for any quasi-isomorphism $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$, by Lemma 10.5(2) $\sigma_{>n}(u)$ is also a quasi-isomorphism. Thus by Proposition 8.10 we get a well-defined functor $\sigma_{>n}$: $D(\mathscr{A}) \to D^+(\mathscr{A})$. Similarly, we get a well-defined functor $\sigma_{n} : D(\mathscr{A}) \to D^-(\mathscr{A})$.

Proposition 10.9. *The canonical functor* $D^+(\mathscr{A}) \to D(\mathscr{A})$ *is fully faithful, so that* $D^+(\mathscr{A})$ *can be identified with the full triangulated subcategory of* $D(\mathscr{A})$ *consisting of* $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ *with bounded below cohomology, i.e.,* $H^n(X^\bullet) = 0$ *for* $n \ll 0$.

Proof. For any quasi-isomorphism $Y^\bullet \to Y'^\bullet$ in $K(\mathscr{A})$ with $Y^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$, by Lemma 10.6 we have a quasi-isomorphism $Y'^\bullet \to \sigma_{>n}(Y'^\bullet)$ with $\sigma_{>n}(Y'^\bullet) \in \mathrm{Ob}(K^+(\mathscr{A}))$ for some $n \in \mathbb{Z}$. Thus Proposition 8.17(1) applies.

Proposition 10.10 (Dual of Proposition 10.9). *The canonical functor* $D^-(\mathscr{A}) \to D(\mathscr{A})$ *is fully faithful, so that* $D^-(\mathscr{A})$ *can be identified with the full triangulated subcategory of* $D(\mathscr{A})$ *consisting of* $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ *with bounded above cohomology, i.e.,* $H^n(X^\bullet) = 0$ *for* $n \gg 0$.

Proposition 10.11. *For* $* = +$ *or* $-$, *the following hold.*
(1) *The canonical functor* $D^{*,\mathrm{b}}(\mathscr{A}) \to D^*(\mathscr{A})$ *is fully faithful.*
(2) *The canonical functor* $D^{\mathrm{b}}(\mathscr{A}) \to D^{*,\mathrm{b}}(\mathscr{A})$ *is an equivalence.*

Proof. Similar to Propositions 10.9 and 10.10.

Definition 10.4. According to Proposition 10.11, we identify each of $D^{\mathrm{b}}(\mathscr{A})$, $D^{+,\mathrm{b}}(\mathscr{A})$ and $D^{-,\mathrm{b}}(\mathscr{A})$ with the full triangulated subcategory of $D(\mathscr{A})$ consisting of complexes with bounded cohomology.

Proposition 10.12. *The canonical functor* $K(\mathscr{A}) \to D(\mathscr{A})$ *induces an isomorphism*

$$K(\mathscr{A})(X^\bullet, I^\bullet) \xrightarrow{\sim} D(\mathscr{A})(X^\bullet, I^\bullet)$$

for all $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *and* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

Proof. By Lemma 4.4 and Proposition 9.13(2).

Proposition 10.13. *Assume* $\mathscr{A}$ *has enough injectives. Then the following hold.*
(1) $K^+(\mathscr{I})$, $K^{+,\mathrm{b}}(\mathscr{I})$ *are full triangulated subcategories of* $K^+(\mathscr{A})$.
(2) *The canonical functor* $K^+(\mathscr{I}) \to D^+(\mathscr{A})$ *is an equivalence of triangulated categories.*
(3) *The canonical functor* $K^{+,\mathrm{b}}(\mathscr{I}) \to D^{+,\mathrm{b}}(\mathscr{A})$ *is an equivalence of triangulated categories.*

Proof. (1) By Proposition 6.1(2).

(2) The canonical functor $K^+(\mathscr{I}) \to D^+(\mathscr{A})$ is fully faithful by Proposition 10.12 and dense by Proposition 4.7.

(3) Similar to (2).

Proposition 10.14 (Dual of Proposition 10.12). *The canonical functor $K(\mathscr{A}) \to D(\mathscr{A})$ induces an isomorphism*

$$K(\mathscr{A})(P^\bullet, Y^\bullet) \tilde{\to} D(\mathscr{A})(P^\bullet, Y^\bullet)$$

for all $P^\bullet \in \operatorname{Ob}(K^-(\mathscr{P}))$ *and* $Y^\bullet \in \operatorname{Ob}(K(\mathscr{A}))$.

Proposition 10.15 (Dual of Proposition 10.13). *Assume $\mathscr{A}$ has enough projectives. Then the following hold.*

(1) $K^-(\mathscr{P})$, $K^{-,\mathrm{b}}(\mathscr{P})$ *are full triangulated subcategories of* $K^-(\mathscr{A})$.

(2) *The canonical functor* $K^-(\mathscr{P}) \to D^-(\mathscr{A})$ *is an equivalence of triangulated categories*.

(2) *The canonical functor* $K^{-,\mathrm{b}}(\mathscr{P}) \to D^{-,\mathrm{b}}(\mathscr{A})$ *is an equivalence of triangulated categories*.

Definition 10.4. A thick subcategory $\mathscr{A}'$ of $\mathscr{A}$ is an abelian exact full subcategory of $\mathscr{A}$ which is closed under extensions.

Remark 10.2. For a ring A the following hold.

(1) The coherent left A-modules form a thick subcategory of Mod A. In case A is left coherent (resp. left noetherian), a left A-module X is coherent if and only if it is finitely presented (resp. finitely generated).

(2) For any two-sided ideal $\mathfrak{a}$ of A,

$$\bigcup_{n\geq 1} \operatorname{Mod} A/\mathfrak{a}^n = \{X \in \operatorname{Mod} A \mid \mathfrak{a}^n X = 0 \text{ for some } n \geq 1\}$$

is a thick subcategory of Mod A.

(3) For a two-sided ideal $\mathfrak{a}$ of A,

$$\operatorname{Mod} A/\mathfrak{a} = \{X \in \operatorname{Mod} A \mid \mathfrak{a} X = 0\}$$

is a thick subcategory of Mod A if and only if $\mathfrak{a}$ is idempotent, i.e., $\mathfrak{a}^2 = \mathfrak{a}$.

(4) For each $X \in \operatorname{Mod} A$ we set

$$r(X) = \{x \in X \mid Ax \text{ has finite length}\}.$$

Then $r : \operatorname{Mod} A \to \operatorname{Mod} A$ is a subfunctor of the identity functor $\mathbf{1}_{\operatorname{Mod} A}$. In case A is left noetherian, the modules $X \in$ Mod A with $X = r(X)$ form a thick subcategory of Mod A.

Definition 10.5. Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. For * = +, –, b or nothing, we denote by $K^*_{\mathscr{A}'}(\mathscr{A})$ the full subcategory of $K^*(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ with $H^n(X^\bullet) \in \mathrm{Ob}(\mathscr{A}')$ for all $n \in \mathbb{Z}$.

Remark 10.3. (1) In case $\mathscr{A}' = \mathscr{A}$, we have $K^*_{\mathscr{A}'}(\mathscr{A}) = K^*(\mathscr{A})$.

(2) In case $\mathscr{A}' = \{0\}$, we have $K^*_{\mathscr{A}'}(\mathscr{A}) = \mathscr{U} \cap K^*(\mathscr{A})$.

Lemma 10.16. *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. Then, for * = +, –,* b *or nothing, the following hold.*

(1) $K^*_{\mathscr{A}'}(\mathscr{A})$ *is a full triangulated subcategory of* $K^*(\mathscr{A})$.

(2) $\mathscr{U} \cap K^*(\mathscr{A})$ *is an épaisse subcategory of* $K^*_{\mathscr{A}'}(\mathscr{A})$.

(3) *If $u : X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $K^*(\mathscr{A})$, then $X^\bullet \in \mathrm{Ob}(K^*_{\mathscr{A}'}(\mathscr{A}))$ if and only if $Y^\bullet \in \mathrm{Ob}(K^*_{\mathscr{A}'}(\mathscr{A}))$.*

Proof. (1) Let $u : X^\bullet \to Y^\bullet$ with $X^\bullet, Y^\bullet \in \mathrm{Ob}(K^*_{\mathscr{A}'}(\mathscr{A}))$. Then, for any $n \in \mathbb{Z}$, since by Proposition 2.4 we have an exact sequence of the form

$$0 \to \mathrm{Cok}\, H^n(u) \to H^n(C(u)) \to \mathrm{Ker}\, H^{n+1}(u) \to 0,$$

it follows that $H^n(C(u)) \in \mathrm{Ob}(K^*_{\mathscr{A}'}(\mathscr{A}))$.

(2) It is obvious that $\mathscr{U} \cap K^*(\mathscr{A}) \subset K^*_{\mathscr{A}'}(\mathscr{A})$. Thus by Lemma 10.1(1) $\mathscr{U} \cap K^*(\mathscr{A})$ is an épaisse subcategory of $K^*_{\mathscr{A}'}(\mathscr{A})$.

(3) Obvious.

Definition 10.6. Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. For * = +, –, b or nothing, according to Lemma 10.16, we have a derived category

$$D^*_{\mathscr{A}'}(\mathscr{A}) = K^*_{\mathscr{A}'}(\mathscr{A})/\mathscr{U} \cap K^*(\mathscr{A}).$$

Proposition 10.17. *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. For * = +, –,* b *or nothing, the canonical functor $D^*_{\mathscr{A}'}(\mathscr{A}) \to D^*(\mathscr{A})$ is fully faithful, so that $D^*_{\mathscr{A}'}(\mathscr{A})$ can be identified with the full triangulated subcategory of $D^*(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$ with $H^n(X^\bullet) \in \mathrm{Ob}(\mathscr{A}')$ for all $n \in \mathbb{Z}$.*

Proof By Lemma 10.16(3) and Proposition 8.17.

Definition 10.7. Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. We denote by $\mathscr{A}' \cap \mathscr{I}$ the collection of objects Ob($\mathscr{A}$') $\cap$ $\mathscr{I}$. Then $\mathscr{A}$' is said to have enough $\mathscr{A}$-injectives if every $X \in$ Ob($\mathscr{A}$') can be embedded in some $I \in \mathscr{A}' \cap \mathscr{I}$.

Lemma 10.18. *Let $\mathscr{A}$' be a thick subcategory of $\mathscr{A}$. Assume $\mathscr{A}$' has enough $\mathscr{A}$-injectives. Then for any* $X^\bullet \in \mathrm{Ob}(K^+_{\mathscr{A}'}(\mathscr{A}))$ *there exists a quasi-isomorphism* $u: X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}' \cap \mathscr{I}))$.

Proof. We may assume $X^n = 0$ for $n < 0$. Put $Z^0 = Z^0(X^\bullet) = H^0(X^\bullet) \in \mathrm{Ob}(\mathscr{A}')$ and let $v^0 : Z^0(X^\bullet) \to Z^0$ be the identity. The following Claim enables us to make use of induction to construct a desired morphism $u: X^\bullet \to I^\bullet$.

Claim: Let $n \geq 0$ and $v^n : Z^n(X^\bullet) \to Z^n$ with $Z^n \in \mathrm{Ob}(\mathscr{A}')$. Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & Z^n(X^\bullet) & \to & X^n & \to & Z^{n+1}(X^\bullet) & \to & H^{n+1}(X^\bullet) & \to & 0 \\ & & {\scriptstyle v^n}\downarrow & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle v^{n+1}} & & \| & & \\ 0 & \to & Z^n & \to & I^n & \to & Z^{n+1} & \to & H^{n+1}(X^\bullet) & \to & 0 \end{array}$$

with $I^n \in \mathscr{A}' \cap \mathscr{I}$ and $Z^{n+1} \in \mathrm{Ob}(\mathscr{A}')$.

Proof. Since Z^n embeds in some $I^n \in \mathscr{A}' \cap \mathscr{I}$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & Z^n(X^\bullet) & \to & X^n & \to & B^{n+1}(X^\bullet) & \to & 0 \\ & & {\scriptstyle v^n}\downarrow & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle w^{n+1}} & & \\ 0 & \to & Z^n & \to & I^n & \to & B^{n+1} & \to & 0, \end{array}$$

with which splice the following push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & B^{n+1}(X^\bullet) & \to & Z^{n+1}(X^\bullet) & \to & H^{n+1}(X^\bullet) & \to & 0 \\ & & {\scriptstyle w^{n+1}}\downarrow & \mathrm{PO} & \downarrow{\scriptstyle v^{n+1}} & & \| & & \\ 0 & \to & B^{n+1} & \to & Z^{n+1} & \to & H^{n+1}(X^\bullet) & \to & 0. \end{array}$$

Proposition 10.19. *Let $\mathscr{A}$' be a thick subcategory of $\mathscr{A}$. Assume $\mathscr{A}$' has enough $\mathscr{A}$-injectives. Then $K^+(\mathscr{A}' \cap \mathscr{I})$ is a full triangulated subcategory of $K^+_{\mathscr{A}'}(\mathscr{A})$ and the canonical functor $K^+(\mathscr{A}' \cap \mathscr{I}) \to D^+_{\mathscr{A}'}(\mathscr{A})$ is an equivalence.*

Proof. By Proposition 6.1(2) $K^{+}(\mathscr{A}' \cap \mathscr{I})$ is a full triangulated subcategory of $K^{+}_{\mathscr{A}'}(\mathscr{A})$. The canonical functor $K^{+}(\mathscr{A}' \cap \mathscr{I}) \to K^{+}_{\mathscr{A}'}(\mathscr{A})$ is fully faithful by Proposition 10.12 and dense by Lemma 10.18.

Definition 10.8. Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. We denote by $\mathscr{A}' \cap \mathscr{P}$ the collection of objects $\mathrm{Ob}(\mathscr{A}') \cap \mathscr{P}$. Then $\mathscr{A}'$ is said to have enough $\mathscr{A}$-projectives if every $X \in \mathrm{Ob}(\mathscr{A}')$ is an epimorph of some $P \in \mathscr{A}' \cap \mathscr{P}$.

Lemma 10.20 (Dual of Lemma 10.18). *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. Assume $\mathscr{A}'$ has enough $\mathscr{A}$-projectives. Then, for any $X^{\bullet} \in \mathrm{Ob}(K^{-}_{\mathscr{A}'}(\mathscr{A}))$, there exists a quasi-isomorphism $u : P^{\bullet} \to X^{\bullet}$ with $P^{\bullet} \in \mathrm{Ob}(K^{-}(\mathscr{A}' \cap \mathscr{P}))$.*

Proposition 10.21 (Dual of Proposition 10.19). *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$. Assume $\mathscr{A}'$ has enough $\mathscr{A}$-projectives. Then $K^{-}(\mathscr{A}' \cap \mathscr{P})$ is a full triangulated subcategory of $K^{-}_{\mathscr{A}'}(\mathscr{A})$ and the canonical functor $K^{-}(\mathscr{A}' \cap \mathscr{P}) \to D^{-}_{\mathscr{A}'}(\mathscr{A})$ is an equivalence.*

Proposition 10.22. *Assume $\mathscr{A}$ satisfies the condition Ab4*. Then the canonical functors $C(\mathscr{A}) \to K(\mathscr{A})$ and $K(\mathscr{A}) \to D(\mathscr{A})$ preserve direct products. In particular, both $K(\mathscr{A})$ and $D(\mathscr{A})$ have arbitrary direct products which are direct products of complexes.*

Proof. By Propositions 1.11(2), 3.4(2) and 9.14.

Proposition 10.23 (Dual of Proposition 10.22). *Assume $\mathscr{A}$ satisfies the condition Ab4. Then the canonical functors $C(\mathscr{A}) \to K(\mathscr{A})$ and $K(\mathscr{A}) \to D(\mathscr{A})$ preserve direct sums. In particular, both $K(\mathscr{A})$ and $D(\mathscr{A})$ have arbitrary direct sums which are direct sums of complexes.*

§11. Hyper Ext

Throughout this section, $\mathscr{A}$ is an abelian category, $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$ and $\mathscr{U}$ is the épaisse subcategory of $K(\mathscr{A})$ consisting of acyclic complexes.

Definition 11.1. For $X^\bullet,\ Y^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ and $n \in \mathbb{Z}$, we set

$$\mathrm{Ext}^n(X^\bullet,\ Y^\bullet) = D(\mathscr{A})(X^\bullet,\ T^n Y^\bullet),$$

which is called the n^{th} hyper Ext.

Proposition 11.1. *Let* $0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$ *be an exact sequence in* $C(\mathscr{A})$ *and put* $\varepsilon = [1 \ \ 0] : C(u) \to T X^\bullet$, $\hat{u} = {}^{t}[u \quad 0] : X^\bullet \to T^{-1}C(v)$ *and* $\hat{v} = [0 \quad v] : C(u) \to Z^\bullet$. *Then the following hold*.

(1) $Q(\hat{u}) : X^\bullet \xrightarrow{\sim} T^{-1}C(v)$ *and* $Q(\hat{v}) : C(u) \xrightarrow{\sim} Z^\bullet$ *are isomorphisms in* $D(\mathscr{A})$.

(2) $(X^\bullet,\ Y^\bullet,\ Z^\bullet, u, v, w)$ *is a triangle in* $D(\mathscr{A})$, *where* $w = Q(\varepsilon) \circ Q(\hat{v})^{-1}$.

(3) *For any* $W^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *we have long exact sequences*

$$\cdots \to \mathrm{Ext}^n(W^\bullet,\ X^\bullet) \to \mathrm{Ext}^n(W^\bullet,\ Y^\bullet) \to \mathrm{Ext}^n(W^\bullet,\ Z^\bullet) \to \mathrm{Ext}^{n+1}(W^\bullet,\ X^\bullet) \to \cdots ,$$

$$\cdots \to \mathrm{Ext}^n(Z^\bullet,\ W^\bullet) \to \mathrm{Ext}^n(Y^\bullet,\ W^\bullet) \to \mathrm{Ext}^n(X^\bullet,\ W^\bullet) \to \mathrm{Ext}^{n+1}(Z^\bullet,\ W^\bullet) \to \cdots .$$

Proof. (1) By Proposition 4.3.

(2) By the part (1) and Proposition 2.5.

(3) By the part (2) and Proposition 6.5.

Definition 11.2. For each $n \in \mathbb{Z}$, we define truncation functors $\sigma'_{\geq n},\ \sigma'_{<n} : C(\mathscr{A}) \to C(\mathscr{A})$ as follows:

$$\sigma'_{\geq n}(X^\bullet)^i = \begin{cases} X^i & (i > n) \\ Z'^n(X^\bullet) & (i = n), \\ 0 & (i < n) \end{cases} \quad \sigma'_{<n}(X^\bullet)^i = \begin{cases} 0 & (i > n) \\ B^n(X^\bullet) & (i = n) \\ X^i & (i < n) \end{cases}$$

for $X^\bullet \in C(\mathscr{A})$. We set $\sigma'_{>n} = \sigma'_{\geq n+1}$ and $\sigma'_{\leq n} = \sigma'_{<n+1}$.

Lemma 11.2 (cf. Lemma 10.5). *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following hold*.

(1) *There exists a natural exact sequence* $0 \to \sigma'_{<n}(X^\bullet) \to X^\bullet \to \sigma'_{\geq n}(X^\bullet) \to 0$.

(2) $$H^i(\sigma'_{\geq n}(X^\bullet)) = \begin{cases} H^i(X^\bullet) & (i \geq n) \\ 0 & (i < n) \end{cases}.$$

(3) $$H^i(\sigma'_{< n}(X^\bullet)) = \begin{cases} 0 & (i \geq n) \\ H^i(X^\bullet) & (i < n) \end{cases}.$$

Lemma 11.3 (cf. Lemma 10.6). *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) *The canonical epimorphism* $X^\bullet \to \sigma'_{\geq n}(X^\bullet)$ *is a quasi-isomorphism.*

(2) $\sigma'_{< n}(X^\bullet)$ *is acyclic.*

(3) $H^i(X^\bullet) = 0$ *for* $i < n$.

Lemma 11.4 (cf. Lemma 10.7). *For any* $n \in \mathbb{Z}$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following are equivalent.*

(1) *The canonical monomorphism* $\sigma'_{< n}(X^\bullet) \to X^\bullet$ *is a quasi-isomorphism.*

(2) $\sigma'_{\geq n}(X^\bullet)$ *is acyclic.*

(3) $H^i(X^\bullet) = 0$ *for* $i \geq n$.

Lemma 11.5 (cf. Lemma 10.8). *For any* $n \in \mathbb{Z}$ *we have truncation functors*

$$\sigma'_{\geq n} : D(\mathscr{A}) \to D^+(\mathscr{A}), \quad \sigma'_{< n} : D(\mathscr{A}) \to D^-(\mathscr{A}).$$

Lemma 11.6. *For* $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ *the following hold.*

(1) *Let* $n \in \mathbb{Z}$ *and assume* $H^i(X^\bullet) = 0$ *for* $i \quad n$. *Then there exist sequences of quasi-isomorphisms*

$$X^\bullet \leftarrow \sigma_{\ n}(X^\bullet) \to \sigma_{\geq n}(\sigma_{\ n}(X^\bullet)) \to T^{-n}(H^n(X^\bullet)),$$
$$X^\bullet \to \sigma'_{\geq n}(X^\bullet) \leftarrow \sigma'_{\leq n}(\sigma'_{\geq n}(X^\bullet)) \leftarrow T^{-n}(H^n(X^\bullet)).$$

(2) *Let* $n, m \in \mathbb{Z}$ *with* $n > m$ *and assume* $H^i(X^\bullet) = 0$ *for* $i > n$ *and* $i < m$. *Then there exist sequences of quasi-isomorphisms*

$$X^\bullet \leftarrow \sigma_{\ n}(X^\bullet) \to \sigma'_{\geq m}(\sigma_{\ n}(X^\bullet)),$$
$$X^\bullet \to \sigma'_{\geq m}(X^\bullet) \leftarrow \sigma_{\ n}(\sigma'_{\geq m}(X^\bullet)).$$

Proof. (1) By Lemmas 10.5, 10.6 and 10.7 we get the first sequence. Also, by Lemmas 11.2, 11.3 and 11.4 we get the last sequence.

(2) By Lemmas 10.5, 10.7 and 11.4 we get the first sequence. Also, by Lemmas 11.2, 11.3 and 10.7 we get the last sequence.

Proposition 11.7. *The canonical functor $\mathscr{A} \to D(\mathscr{A})$ induces an equivalence between $\mathscr{A}$ and the full subcategory of $D(\mathscr{A})$ consisting of $X^\bullet$ with $H^i(X^\bullet) = 0$ for $i \quad 0$.*

Proof. Let $X, Y \in \mathrm{Ob}(\mathscr{A})$. Let $J : \mathscr{A} \to D(\mathscr{A})$ denote the canonical functor. Then $H^0 \circ J = \mathbf{1}_{\mathscr{A}}$ and $H^0 : D(\mathscr{A})(X, Y) \to \mathscr{A}(X, Y)$ is epic. Let $u \in D(\mathscr{A})(X, Y)$ with $H^0(u) = 0$. We claim $u = 0$. Let $u = Q(s)^{-1}Q(f)$ with $s : Y \to Y^\bullet$ a quasi-isomorphism. By Lemma 11.3(1) the canonical epimorphism $t : Y^\bullet \to \sigma'_{\geq 0}(Y^\bullet)$ is a quasi-isomorphism. Also, since $Q(f) = Q(s)u$, $H^0(f) = H^0(s) \circ H^0(u) = 0$. Thus $f : X \to Y^0$ factors through $B^0(Y^\bullet)$ and $tf = 0$. It follows that $u = Q(s)^{-1}Q(f) = Q(ts)^{-1}Q(tf) = 0$. Hence $H^0 : D(\mathscr{A})(X, Y) \to \mathscr{A}(X, Y)$ is an isomorphism, so is $J : \mathscr{A}(X, Y) \to D(\mathscr{A})(X, Y)$. The last assertion follows by Lemma 11.6(1).

Definition 11.3. Let $X, Y \in \mathrm{Ob}(\mathscr{A})$ and $n \geq 1$. An n-extension of X by Y is an exact sequence in $\mathscr{A}$ of the form

$$E:\ 0 \to Y \to E^{-n+1} \to \cdots \to E^0 \to X \to 0.$$

For two n-extensions E and E', we define a homomorphism $f : E \to E'$ as a family of morphisms $f^i : E^i \to E'^i$ $(-n + 1 \leq i \leq 0)$ in $\mathscr{A}$ which make the following diagram commute

$$\begin{array}{ccccccccccccc}
E:\ 0 & \to & Y & \to & E^{-n+1} & \to & \cdots & \to & E^0 & \to & X & \to & 0 \\
 & & \| & & \downarrow f^{-n+1} & & & & \downarrow f^0 & & \| & & \\
E':\ 0 & \to & Y & \to & E'^{-n+1} & \to & \cdots & \to & E'^0 & \to & X & \to & 0
\end{array}$$

and denote by $\mathrm{Hom}(E, E')$ the set of homomorphisms from E to E'. An equivalence relation $\sim$ on the collection of n-extensions is defined as follws: $E \sim E'$ if and only if there exists a sequence of n-extensions $E_0 = E, \cdots, E_k = E'$ such that $\mathrm{Hom}(E_i, E_{i+1}) \cup \mathrm{Hom}(E_{i+1}, E_i) \quad \varnothing$ for all $0 \leq i \leq k - 1$. We denote by $[E]$ the equivalence class of an n-extension E and by $\mathrm{Ext}^n_{\mathscr{A}}(X, Y)$ the collection of equivalence classes.

Definition 11.4. Let $X, Y \in \mathrm{Ob}(\mathscr{A})$ and $n \geq 1$. For each n-extension

$$E:\ 0 \to Y \xrightarrow{\mu} E^{-n+1} \to \cdots \to E^0 \xrightarrow{\varepsilon} X \to 0,$$

we denote by $E^\bullet$ the complex

$$\cdots \to 0 \to E^{-n+1} \to \cdots \to E^0 \to 0 \to \cdots .$$

Then we have homomorphisms of complexes $\mu : T^{n-1}Y \to E^\bullet$, $\varepsilon : E^\bullet \to X$. The mapping cone $C(\mu)$ is of the form

$$\cdots \to 0 \to Y \xrightarrow{\mu} E^{-n+1} \to \cdots \to E^0 \to 0 \to \cdots$$

and we have a quasi-isomorphism $\hat{\varepsilon} = [0 \quad \varepsilon] : C(\mu) \to X$. Thus we get a commutative diagram in $D(\mathscr{A})$

$$\begin{array}{ccccccc} T^{n-1}Y & \xrightarrow{\mu} & E^\bullet & \xrightarrow{\alpha} & C(\mu) & \xrightarrow{\beta} & T^nY \\ \| & & \| & & \downarrow \varepsilon & & \| \\ T^{n-1}Y & \xrightarrow{\mu} & E^\bullet & \xrightarrow{\varepsilon} & X & \xrightarrow{u(E)} & T^nY \end{array}$$

with the top row a triangle, where $\alpha = {}^t[0 \quad 1]$ and $\beta = [1 \quad 0]$. Hence the bottom row is also a triangle.

Definition 11.5. Let $X, Y \in \mathrm{Ob}(\mathscr{A})$ and $n \geq 1$. Embed each $u \in D(\mathscr{A})(X, T^nY)$ in a triangle $(T^{n-1}Y, E^\bullet, X, \cdot, \cdot, u)$. In case $n = 1$, the long exact sequence

$$\cdots \to H^{-1}(X) \to H^0(Y) \to H^0(E^\bullet) \to H^0(Y) \to H^1(X) \to \cdots$$

yields a 1-extension

$$E(u) : 0 \to Y \to H^0(E^\bullet) \to X \to 0,$$

and in case $n \geq 2$, since

$$H^i(E^\bullet) = \begin{cases} X & (i = 0) \\ Y & (i = -n+1), \\ 0 & \text{otherwise} \end{cases}$$

we get an n-extension

$$E(u) : 0 \to Y \to Z'^{-n+1}(E^\bullet) \to E^{-n+2} \to \cdots \to E^{-1} \to Z^0(E^\bullet) \to X \to 0.$$

Proposition 11.8. *For any $n \geq 1$ and $X, Y \in \mathrm{Ob}(\mathscr{A})$ we have a natural isomorphism*

$$\phi : \mathrm{Ext}^n_{\mathscr{A}}(X, Y) \xrightarrow{\sim} D(\mathscr{A})(X, T^nY),\ [E] \mapsto u(E),$$

whose inverse is given by

$$\psi : D(\mathscr{A})(X, T^nY) \xrightarrow{\sim} \operatorname{Ext}^n_{\mathscr{A}}(X, Y),\ u \mapsto [E(u)].$$

Proof. We divide the proof into several steps.

Claim 1: ϕ is well defined.

Proof. Let

$$\begin{array}{ccccccccccccc} E: & 0 & \to & Y & \xrightarrow{\mu} & E^{-n+1} & \to & \cdots & \to & E^0 & \xrightarrow{\varepsilon} & X & \to & 0 \\ & & & \| & & \downarrow f^{-n+1} & & & & \downarrow f^0 & & \| & & \\ E': & 0 & \to & Y & \xrightarrow{\mu'} & E'^{-n+1} & \to & \cdots & \to & E'^0 & \xrightarrow{\varepsilon'} & X & \to & 0 \end{array}$$

be a homomorphism of n-extensions. Denote by $\hat{f} : C(\mu) \to C(\mu')$ the homomorphism of complexes

$$\begin{array}{ccccccccccccc} \cdots & \to & 0 & \to & Y & \xrightarrow{\mu} & E^{-n+1} & \to & \cdots & \to & E^0 & \to & 0 & \to & \cdots \\ & & \downarrow & & \| & & \downarrow f^{-n+1} & & & & \downarrow f^0 & & \downarrow & & \\ \cdots & \to & 0 & \to & Y & \xrightarrow{\mu'} & E'^{-n+1} & \to & \cdots & \to & E'^0 & \to & 0 & \to & \cdots . \end{array}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\varepsilon} & E^\bullet & \to & T^nY \\ \| & & \downarrow \hat{f} & & \| \\ X & \xleftarrow{\varepsilon'} & E'^\bullet & \to & T^nY. \end{array}$$

It follows that $u(E) = u(E')$.

Claim 2: ψ is well defined.

Proof. Let $u \in D(\mathscr{A})(X, T^nY)$. Embed u in triangles

$$(T^{n-1}Y,\ E^\bullet, X, \cdot, \cdot, u) \quad \text{and} \quad (T^{n-1}Y,\ E'^\bullet, X, \cdot, \cdot, u).$$

Then we have an isomorphism of triangles

$$\begin{array}{ccccccc} T^{n-1}Y & \to & E^\bullet & \to & X & \xrightarrow{u} & T^nY \\ \| & & \downarrow f & & \| & & \| \\ T^{n-1}Y & \to & E'^\bullet & \to & X & \xrightarrow{u} & T^nY\,. \end{array}$$

In case $n = 1$, we get an isomorphism of 1-extensions

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & H^0(E^\bullet) & \to & X & \to & 0 \\ & & \| & & \downarrow H^0(f) & & \| & & \\ 0 & \to & Y & \to & H^0(E'^\bullet) & \to & X & \to & 0\,, \end{array}$$

and in case $n \geq 2$, we get an isomorphism of n-extensions

$$\begin{array}{ccccccccccccccc} 0 & \to & Y & \to & Z'^{-n+1}(E^\bullet) & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & Z^0(E^\bullet) & \to & X \to 0 \\ & & \| & & \downarrow Z'^{-n+1}(f) & & \downarrow f^{-n+2} & & & & \downarrow f^{-1} & & \downarrow Z^0(f) & & \| \\ 0 & \to & Y & \to & Z'^{-n+1}(E'^\bullet) & \to & E'^{-n+2} & \to & \cdots & \to & E'^{-1} & \to & Z^0(E'^\bullet) & \to & X \to 0\,. \end{array}$$

Claim 3: $\psi \circ \phi = \mathrm{id}$.

Proof. Let

$$E:\ 0 \xrightarrow{\mu} Y \to E^{-n+1} \to \cdots \to E^0 \xrightarrow{\varepsilon} X \to 0$$

be an n-extension. Let $(T^{n-1}Y,\ E^\bullet, X, \mu, \varepsilon, u(E))$ be a triangle associated with E. Then, since $\sigma'_{>-n}(\sigma_n(E^\bullet)) = E^\bullet$, it follows that $E(u(E)) = E$.

Claim 4: $\phi \circ \psi = \mathrm{id}$.

Proof. Let $u \in D(\mathscr{A})(X, T^nY)$ and embed it in a triangle $(T^{n-1}Y,\ E^\bullet, X, \cdot, \cdot, u)$. Consider first the case $n = 1$. Then the corresponding 1-extension is of the form

$$E(u):\ 0 \to Y \xrightarrow{\mu} H^0(E^\bullet) \xrightarrow{\varepsilon} X \to 0$$

and by Lemma 11.6 we have a sequence of quasi-isomorphisms

$$E^\bullet \leftarrow \sigma_0(X^\bullet) \to \sigma_{\geq 0}(\sigma_0(X^\bullet)) \to H^0(E^\bullet).$$

Also, we have a commutative diagram

$$\begin{array}{ccccccc}
Y & \to & E^\bullet & \to & X & \overset{u}{\to} & TY \\
\| & & \uparrow & & \| & & \| \\
Y & \to & \sigma_0(X^\bullet) & \to & X & \overset{u}{\to} & TY \\
\| & & \downarrow & & \| & & \| \\
Y & \to & \sigma_{\geq 0}(\sigma_0(X^\bullet)) & \to & X & \overset{u}{\to} & TY \\
\| & & \downarrow & & \| & & \| \\
Y & \overset{\mu}{\to} & H^0(E^\bullet) & \overset{\varepsilon}{\to} & X & \overset{u}{\to} & TY.
\end{array}$$

Thus the bottom row is a triangle and $u = u(E(u))$. Next, let $n \geq 2$. Then the corresponding n-extension is of the form

$$E(u):\ 0 \to Y \overset{\mu}{\to} Z'^{-n+1}(E^\bullet) \to E^{-n+2} \to \cdots \to E^{-1} \to Z^0(E^\bullet) \overset{\varepsilon}{\to} X \to 0.$$

Note that $E(u)^\bullet = \sigma'_{>-n}(\sigma_0(X^\bullet))$. Thus by Lemma 11.6 we have a sequence of quasi-isomorphisms $E^\bullet \leftarrow \sigma_0(X^\bullet) \to E(u)^\bullet$. Since we have a commutative diagram

$$\begin{array}{ccccccc}
T^{n-1}Y & \to & E^\bullet & \to & X & \overset{u}{\to} & T^nY \\
\| & & \uparrow & & \| & & \| \\
T^{n-1}Y & \to & \sigma_0(X^\bullet) & \to & X & \overset{u}{\to} & T^nY \\
\| & & \downarrow & & \| & & \| \\
T^{n-1}Y & \overset{\mu}{\to} & E(u)^\bullet & \overset{\varepsilon}{\to} & X & \overset{u}{\to} & T^nY,
\end{array}$$

the bottom row is a triangle and $u = u(E(u))$.

Claim 5: ϕ is natural.

Proof. Let

$$E:\ 0 \to Y \overset{\mu}{\to} E^{-n+1} \to \cdots \to E^0 \overset{\varepsilon}{\to} X \to 0,$$

be an n-extension and $f \in \mathscr{A}(X', X)$, $g \in \mathscr{A}(Y, Y')$. Take a pull-back and a push-out successively

$$\begin{array}{ccccccccccccccc}
E: & 0 & \to & Y & \xrightarrow{\mu} & E^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E^{0} & \xrightarrow{\varepsilon} & X & \to & 0 \\
 & & & \| & & \| & & \| & & & & \| & & f^0\uparrow & \text{PB} & \uparrow f & & \\
E': & 0 & \to & Y & \xrightarrow{\mu} & E^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E'^{0} & \xrightarrow{\varepsilon'} & X' & \to & 0 \\
 & & & g\downarrow & \text{PO} & \downarrow g^{-n+1} & & \| & & & & \| & & \| & & \| & & \\
E'': & 0 & \to & Y' & \xrightarrow{\mu'} & E'^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E'^{0} & \xrightarrow{\varepsilon'} & X' & \to & 0 .
\end{array}$$

Denote by $\hat{f} : E^{\bullet} \to E'^{\bullet}$ and $\hat{g} : E'^{\bullet} \to E''^{\bullet}$ the homomorphisms of complexes

$$\begin{array}{ccccccccccccccc}
\cdots & \to & 0 & \to & E^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E^{0} & \to & 0 & \to & \cdots \\
 & & \uparrow & & \| & & \| & & & & \| & & \uparrow f^0 & & \uparrow & & \\
\cdots & \to & 0 & \to & E^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E'^{0} & \to & 0 & \to & \cdots,
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\cdots & \to & 0 & \to & E^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E'^{0} & \to & 0 & \to & \cdots \\
 & & \downarrow & & \downarrow g^{-n+1} & & \| & & & & \| & & \| & & \downarrow & & \\
\cdots & \to & 0 & \to & E'^{-n+1} & \to & E^{-n+2} & \to & \cdots & \to & E^{-1} & \to & E'^{0} & \to & 0 & \to & \cdots,
\end{array}$$

respectively. Then we have homomorphisms of triangles

$$\begin{array}{ccccccc}
T^{n-1}Y & \xrightarrow{\mu} & E^{\bullet} & \xrightarrow{\varepsilon} & X & \xrightarrow{u(E)} & T^{n}Y \\
\| & & \uparrow \hat{f} & & \uparrow f & & \| \\
T^{n-1}Y & \xrightarrow{\mu} & E'^{\bullet} & \xrightarrow{\varepsilon'} & X' & \xrightarrow{u(E')} & T^{n}Y \\
T^{n-1}(g)\downarrow & & \downarrow \hat{g} & & \| & & \downarrow T^{n}(g) \\
T^{n-1}Y' & \xrightarrow{\mu'} & E''^{\bullet} & \xrightarrow{\varepsilon'} & X' & \xrightarrow{u(E'')} & T^{n}Y' .
\end{array}$$

Hence $\phi(\operatorname{Ext}^{n}_{\mathscr{A}}(f, g)([E])) = D(\mathscr{A})(f, T^{n}(g))(\phi([E]))$.

Proposition 11.9. *For any $X, Y \in \operatorname{Ob}(\mathscr{A})$ the following hold.*
(1) *If Y has an injective resolution $Y \to I_Y^{\bullet}$, then $\operatorname{Ext}^{i}(X, Y) \cong H^{i}(\mathscr{A}(X, I_Y^{\bullet}))$ for all $i \in \mathbb{Z}$.*
(2) *If X has a projective resolution $P_X^{\bullet} \to X$, then $\operatorname{Ext}^{i}(X, Y) \cong H^{i}(\mathscr{A}(P_X^{\bullet}, Y))$ for all $i \in \mathbb{Z}$.*

Proof. (1) For any $i \in \mathbb{Z}$, by Propositions 10.12 and 3.8 we have

$$\begin{aligned}
\operatorname{Ext}^{i}(X, Y) &\cong D(\mathscr{A})(X, T^{i}(Y)) \\
&\cong D(\mathscr{A})(X, T^{i}(I_Y^{\bullet}))
\end{aligned}$$

$$\cong K(\mathscr{A})(X, T^i(I_Y^\bullet))$$
$$\cong H^i(\mathscr{A}(X, I_Y^\bullet)).$$

(2) Dual of (1).

Definition 11.6. A complex $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ is said to have finite injective dimension if $\mathrm{Ext}^i(-, X^\bullet)$ vanishes on $\mathscr{A}$ for $i \gg 0$. For $* = +, -$, b or nothing, we denote by $K^*(\mathscr{A})_{\mathrm{fid}}$ the full subcategory of $K^*(\mathscr{A})$ consisting of $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ which have finite injective dimension.

Lemma 11.10. *For* $* = +, -$, b *or nothing*, *the following hold.*

(1) $K^*(\mathscr{A})_{\mathrm{fid}}$ *is a full triangulated subcategory of* $K^*(\mathscr{A})$.

(2) $\mathscr{U} \cap K^*(\mathscr{A})_{\mathrm{fid}}$ *is an épaisse subcategory of* $K^*(\mathscr{A})_{\mathrm{fid}}$.

Proof. (1) For any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A})_{\mathrm{fid}})$ and $j \in \mathbb{Z}$, since $\mathrm{Ext}^i(-, T^j(X^\bullet)) \cong \mathrm{Ext}^{i+j}(-, X^\bullet)$ vanishes on $\mathscr{A}$ for $i \gg 0$, $T^j(X^\bullet) \in \mathrm{Ob}(K^*(\mathscr{A})_{\mathrm{fid}})$. Also, for any $u \in K(\mathscr{A})(X^\bullet, Y^\bullet)$ with $X^\bullet$, $Y^\bullet \in \mathrm{Ob}(K^*(\mathscr{A})_{\mathrm{fid}})$, since by Proposition 6.5 we have a long exact sequence

$$\cdots \to \mathrm{Ext}^i(-, Y^\bullet) \to \mathrm{Ext}^i(-, C(u)) \to \mathrm{Ext}^{i+1}(-, X^\bullet) \to \cdots ,$$

$\mathrm{Ext}^i(-, C(u))$ vanishes on $\mathscr{A}$ for $i \gg 0$ and $C(u) \in \mathrm{Ob}(K^*(\mathscr{A})_{\mathrm{fid}})$.

(2) By Proposition 7.7.

Definition 11.7. For $* = +, -$, b or nothing, according to Lemma 11.10, we have a derived category

$$D^*(\mathscr{A})_{\mathrm{fid}} = K^*(\mathscr{A})_{\mathrm{fid}} / \mathscr{U} \cap K^*(\mathscr{A})_{\mathrm{fid}}.$$

Lemma 11.11. *For* $* = +, -$, b *or nothing*, *the canonical functor* $D^*(\mathscr{A})_{\mathrm{fid}} \to D^*(\mathscr{A})$ *is fully faithful*.

Proof. It is obvious that $K^*(\mathscr{A})_{\mathrm{fid}}$ is closed under quasi-isomorphism classes in $K^*(\mathscr{A})$. Thus by Proposition 8.17 the canonical functor $D^*(\mathscr{A})_{\mathrm{fid}} \to D^*(\mathscr{A})$ is fully faithful.

Proposition 11.12. *Assume* $\mathscr{A}$ *has enough injectives. Then for* $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A})_{\mathrm{fid}})$.

(2) *There exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$.

Proof. (1) $\Rightarrow$ (2). Let $n \in \mathbb{Z}$ and assume $\mathrm{Ext}^i(-, X^\bullet)$ vanishes on $\mathscr{A}$ for $i > n$. By

Proposition 4.7 there exists a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathcal{I}))$. Let $i > n$. Let $j : Z^i(I^\bullet) \to I^i$ denote the inclusion. Since by Proposition 10.12 we have

$$\begin{aligned} K(\mathcal{A})(T^{-i}(Z^i(I^\bullet)),\ I^\bullet) &\cong K(\mathcal{A})(Z^i(I^\bullet), T^i(I^\bullet)) \\ &\cong D(\mathcal{A})(Z^i(I^\bullet), T^i(I^\bullet)) \\ &\cong \mathrm{Ext}^i(Z^i(I^\bullet),\ I^\bullet) \\ &= 0, \end{aligned}$$

there exists $h : Z^i(I^\bullet) \to I^{i-1}$ such that $j = d_I^{i-1} \circ h$. Thus $B^i(I^\bullet) = Z^i(I^\bullet)$ and the canonical epimorphism $I^{i-1} \to B^i(I^\bullet)$ splits. Consequently, $H^i(I^\bullet) = 0$ and $Z^{i-1}(I^\bullet) \in \mathcal{I}$ for all $i > n$. Thus $\sigma_n(I^\bullet) \in \mathrm{Ob}(K^{\mathrm{b}}(\mathcal{I}))$ and by Lemma 10.7 the canonical monomorphism $\hat{j} : \sigma_n(I^\bullet) \to I^\bullet$ is a quasi-isomorphism. Then by Corollary 4.6 $\hat{j} : \sigma_n(I^\bullet) \to I^\bullet$ is an isomorphism in $K(\mathcal{A})$ and we get a quasi-isomorphism $\hat{j}^{-1} \circ s : X^\bullet \to \sigma_n(I^\bullet)$.

(2) ⇒ (1). Let $i \in \mathbb{Z}$ with $I^i = 0$. Then for any $Y \in \mathrm{Ob}(\mathcal{A})$ by Propositions 10.12 and 3.8 we have

$$\begin{aligned} \mathrm{Ext}^i(Y,\ X^\bullet) &\cong \mathrm{Ext}^i(Y,\ I^\bullet) \\ &\cong D(\mathcal{A})(Y, T^i(I^\bullet)) \\ &\cong K(\mathcal{A})(Y, T^i(I^\bullet)) \\ &\cong H^i(\mathcal{A}(Y,\ I^\bullet)) \\ &= 0. \end{aligned}$$

Proposition 11.13. *Assume $\mathcal{A}$ has enough injectives. Then the following hold.*

(1) $K^{\mathrm{b}}(\mathcal{I}) \subset K^+(\mathcal{A})_{\mathrm{fid}} \subset K^{+,\mathrm{b}}(\mathcal{A})$.

(2) $K^{\mathrm{b}}(\mathcal{I})$ *is a full triangulated subcategory of* $K^+(\mathcal{A})_{\mathrm{fid}}$.

(3) *The canonical functor* $K^+(\mathcal{A})_{\mathrm{fid}} \to D^+(\mathcal{A})_{\mathrm{fid}}$ *induces an equivalence* $K^{\mathrm{b}}(\mathcal{I}) \xrightarrow{\sim} D^+(\mathcal{A})_{\mathrm{fid}}$.

Proof. (1) By Proposition 11.12.

(2) By Proposition 6.1(2).

(3) The canonical functor $K^{\mathrm{b}}(\mathcal{I}) \to D^+(\mathcal{A})_{\mathrm{fid}}$ is fully faithful by Corollary 4.6 and dense by Proposition 11.12.

Definition 11.8. A complex $X^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$ is said to have finite projective dimension if $\mathrm{Ext}^i(X^\bullet, -)$ vanishes on $\mathcal{A}$ for $i \gg 0$. For * = +, –, b or nothing, we denote by $K^*(\mathcal{A})_{\mathrm{fpd}}$ the full subcategory of $K^*(\mathcal{A})$ consisting of $X^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$ which have finite projective dimension.

Lemma 11.14 (Dual of Lemma 11.10). *For* * = +, –, b *or nothing, the following hold.*

(1) $K^*(\mathcal{A})_{\mathrm{fpd}}$ *is a full triangulated subcategory of* $K^*(\mathcal{A})$.

(2) $\mathcal{U} \cap K^*(\mathcal{A})_{\mathrm{fpd}}$ is an épaisse subcategory of $K^*(\mathcal{A})_{\mathrm{fpd}}$.

Definition 11.9. For * = +, –, b or nothing, according to Lemma 11.14, we have a derived category

$$D^*(\mathscr{A})_{\mathrm{fpd}} = K^*(\mathscr{A})_{\mathrm{fpd}}/\mathscr{U} \cap K^*(\mathscr{A})_{\mathrm{fpd}}.$$

Lemma 11.15 (Dual of Lemma 11.11). *For* * = +, –, b *or nothing, the canonical functor* $D^*(\mathscr{A})_{\mathrm{fpd}} \to D^*(\mathscr{A})$ *is fully faithful.*

Proposition 11.16 (Dual of Proposition 11.12). *Assume* $\mathscr{A}$ *has enough projectives. Then for* $X^\bullet \in \mathrm{Ob}(K^-(\mathscr{A}))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(K^-(\mathscr{A})_{\mathrm{fpd}})$.

(2) *There exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{P}))$.

Proposition 11.17 (Dual of Proposition 11.13). *Assume* $\mathscr{A}$ *has enough injectives. Then the following hold.*

(1) $K^{\mathrm{b}}(\mathscr{P}) \subset K^-(\mathscr{A})_{\mathrm{fpd}} \subset K^{-,\,\mathrm{b}}(\mathscr{A})$.

(2) $K^{\mathrm{b}}(\mathscr{P})$ *is a full triangulated subcategory of* $K^-(\mathscr{A})_{\mathrm{fpd}}$.

(3) *The canonical functor* $K^-(\mathscr{A})_{\mathrm{fpd}} \to D^-(\mathscr{A})_{\mathrm{fpd}}$ *induces an equivalence* $K^{\mathrm{b}}(\mathscr{P}) \xrightarrow{\sim} D^-(\mathscr{A})_{\mathrm{fpd}}$.

Proposition 11.18. *Assume* $\mathscr{A}$ *has enough injectives and satisfies the condition Ab4*. Then the canonical functors* $D^+(\mathscr{A}) \to D(\mathscr{A})$, $D^+(\mathscr{A})_{\mathrm{fid}} \to D^+(\mathscr{A})$ *preserve direct products.*

Proof. Let $\{X_\lambda^\bullet\}_{\lambda \in \Lambda}$ be a family of objects of $D^+(\mathscr{A})$ which has a direct product $X^\bullet$ in $D^+(\mathscr{A})$. Note that by Proposition 10.11 the direct product $\prod X_\lambda^\bullet$ exists in $D(\mathscr{A})$. We claim $\prod X_\lambda^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$. By Proposition 4.7 we may assume the $X_\lambda^\bullet$ and $X^\bullet$ are objects of $K^+(\mathscr{I})$. Take $b \in \mathbb{Z}$ such that $X^i = 0$ for $i < b$. Let $\lambda \in \Lambda$ and put $m = \min\{i \in \mathbb{Z} \mid H^i(X_\lambda^\bullet) \;\; 0\}$. Then by Lemma 11.3 we have a quasi-isomorphism $s : X_\lambda^\bullet \to \sigma'_{\geq m}(X_\lambda^\bullet)$. Also, there exists $u : T^{-m}(H^m(X_\lambda^\bullet)) \to \sigma'_{\geq m}(X_\lambda^\bullet)$ such that $H^m(u) = \mathrm{id}_{H^m(X_\lambda^\bullet)}$. Thus we get a nonzero morphism $Q(s)^{-1} \circ Q(u) \in D(\mathscr{A})(T^{-m}(H^m(X_\lambda^\bullet)), X_\lambda^\bullet)$ and by Proposition 10.12 we get

$$K(\mathscr{A})(T^{-m}(H^m(X_\lambda^\bullet)), X^\bullet) \cong D(\mathscr{A})(T^{-m}(H^m(X_\lambda^\bullet)), X^\bullet) \neq 0.$$

Hence $m \geq b$ and $H^i(X_\lambda^\bullet) = 0$ for $i < b$. Since we have an exact sequence

$$\cdots \to X_\lambda^{b-1} \to X_\lambda^b \to Z'^b(X_\lambda^\bullet) \to 0,$$

$X_\lambda^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ implies $Z'^b(X_\lambda^\bullet) \in \mathscr{I}$ and $\sigma'_{\geq b}(X_\lambda^\bullet) \in \mathrm{Ob}(K^+(\mathscr{I}))$. Thus we may assume X_λ^i

$= 0$ for all $i < b$ and $\lambda \in \Lambda$. Then $\prod X_\lambda^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ and $\prod X_\lambda^\bullet \cong X^\bullet$.

Next, let $\{X_\lambda^\bullet\}_{\lambda \in \Lambda}$ be a family of objects of $D^+(\mathscr{A})_{\mathrm{fid}}$ which has a direct product $X^\bullet$ in $D^+(\mathscr{A})_{\mathrm{fid}}$. We claim $\prod X_\lambda^\bullet \in \mathrm{Ob}(D^+(\mathscr{A})_{\mathrm{fid}})$. By Proposition 11.12 we may assume the $X_\lambda^\bullet$ and $X^\bullet$ are objects of $K^{\mathrm{b}}(\mathscr{I})$. Take $a, b \in \mathbb{Z}$ such that $X^i = 0$ for $i > a$ and $i < b$. As above, we may assume $X_\lambda^i = 0$ for all $i < b$ and $\lambda \in \Lambda$. Let $\lambda \in \Lambda$ and put $n = \max\{i \in \mathbb{Z} \mid H^i(X_\lambda^\bullet) \quad 0\}$. By Lemma 10.6 we have a quasi-isomorphism $t : \sigma_{\ n}(X_\lambda^\bullet) \to X_\lambda^\bullet$. Also, there exists a morphism $v : T^{-n}(Z^n(X_\lambda^\bullet)) \to \sigma_{\ n}(X_\lambda^\bullet)$ such that $H^n(v) : Z^n(X_\lambda^\bullet) \to H^n(X_\lambda^\bullet)$ is the canonical epimorphism. Thus we have a nonzero morphism $Q(tv) \in D(\mathscr{A})(T^{-n}(Z^n(X_\lambda^\bullet)), X_\lambda^\bullet)$. Also, since $Z^n(X_\lambda^\bullet)$ admits an injective resolution

$$0 \to Z^n(X_\lambda^\bullet) \to X_\lambda^n \to X_\lambda^{n+1} \to \cdots,$$

$X_\lambda^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$ implies $T^{-n}(Z^n(X_\lambda^\bullet)) \in \mathrm{Ob}(D^+(\mathscr{A})_{\mathrm{fid}})$. Thus $D(\mathscr{A})(T^{-n}(Z^n(X_\lambda^\bullet)), X^\bullet) \quad 0$ and $n \le a$. Next, since $D(\mathscr{A})(T^{-(a+1)}(B'^a(X_\lambda^\bullet)), X^\bullet) = 0$, by Proposition 10.12 we have

$$\begin{aligned} K(\mathscr{A})(T^{-(a+1)}(B'^a(X_\lambda^\bullet)), X_\lambda^\bullet) &\cong D(\mathscr{A})(T^{-(a+1)}(B'^a(X_\lambda^\bullet)), X_\lambda^\bullet) \\ &= 0 \end{aligned}$$

and the canonical exact sequence $0 \to Z^a(X_\lambda^\bullet) \to X_\lambda^a \to B'^a(X_\lambda^\bullet) \to 0$ splits. Thus we have a quasi-isomorphism $\sigma_{\ a}(X_\lambda^\bullet) \to X_\lambda^\bullet$ with $\sigma_{\ a}(X_\lambda^\bullet) \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$. Consequently, we may assume $X_\lambda^i = 0$ for all $i > a$ and $\lambda \in \Lambda$. Then $\prod X_\lambda^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$ and $\prod X_\lambda^\bullet \cong X^\bullet$.

Proposition 11.19 (Dual of Proposition 11.18). *Assume $\mathscr{A}$ has enough projectives and satisfies the condition Ab4. Then the canonical functors $D^-(\mathscr{A}) \to D(\mathscr{A})$, $D^-(\mathscr{A})_{\mathrm{fpd}} \to D^-(\mathscr{A})$ preserve direct sums.*

Proposition 11.20. *Let $0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$ be an exact sequence in $C(\mathscr{A})$. Assume $Y^\bullet = 0$ in $K(\mathscr{A})$, this is the case if $Y^\bullet$ is either injective or projective in $C(\mathscr{A})$. Then the following hold.*

(1) *There exists $h : u \simeq 0$ and $vh : TX^\bullet \to Z^\bullet$ is a quasi-isomorphism for all $h : u \simeq 0$.*

(2) *There exists $h : v \simeq 0$ and $T^{-1}(h)u : X^\bullet \to T^{-1}Z^\bullet$ is a quasi-isomorphism for all $h : v \simeq 0$.*

Proof. It follows by Propositions 3.5 and 3.6 that $Y^\bullet = 0$ in $K(\mathscr{A})$ if $Y^\bullet$ is either injective or projective in $C(\mathscr{A})$.

(1) Since $u = 0$ in $K(\mathscr{A})$, there exists $h : u \simeq 0$. Next, take an arbitrary $h : u \simeq 0$ and put $\hat{h} = {}^{t}[1 \ \ -h] : TX^\bullet \to C(u)$. Then, since $u = h \circ d_X + T^{-1}(d_Y \circ h)$, $d_{C(u)} \circ \hat{h} = T\hat{h} \circ d_Z$ and $\hat{h}$ is a morphism in $C(\mathscr{A})$. Put $\hat{v} = [0 \ \ v] : C(u) \to Z^\bullet$. Then by Proposition 11.1(1) $Q(\hat{v})$ is an isomorphism in $D(\mathscr{A})$. Let $\varepsilon = [1 \ \ 0] : C(u) \to TX^\bullet$ and $w = Q(\varepsilon) \circ Q(\hat{v})^{-1}$. Then, since by

Proposition 11.1(2) $(X^\bullet, Y^\bullet, Z^\bullet, u, v, w)$ is a triangle in $D(\mathscr{A})$, and since $Y^\bullet = 0$ in $D(\mathscr{A})$, by Lemma 6.9 w is an isomorphism in $D(\mathscr{A})$. Thus $Q(\varepsilon)$ is an isomorphism, so is $Q(\hat{h})$ because $\varepsilon \circ \hat{h} = \mathrm{id}_{TX}$. Hence $Q(\hat{v}\hat{h})$ is an isomorphism and by Proposition 10 3 $vh = -\hat{v}\hat{h}$ is a quasi-isomorphism.

(2) Dual of (1).

§12. Localization in triangulated categories

Throughout this section, $\mathcal{K}$ and $\mathcal{H}$ are triangulated categories. Also, $\mathcal{A}$ is an abelian category, $\mathcal{U}$ is the épaisse subcategory of $K(\mathcal{A})$ consisting of acyclic complexes and $\mathcal{I}$ (resp. $\mathcal{P}$) is the collection of injective (resp. projective) objects of $\mathcal{A}$. Unless otherwise stated, functors are covariant functors.

Proposition 12.1. *Let $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ be a ∂-functor. Assume F has a right adjoint $G : \mathcal{H} \to \mathcal{K}$. Let $\varepsilon : \mathbf{1}_{\mathcal{K}} \to GF$, $\delta : FG \to \mathbf{1}_{\mathcal{H}}$ be the unit and the counit, respectively, and put $\eta = GT\delta \circ G\theta_G \circ \varepsilon_{TG} : TG \to GT$. Then the following hold.*

(1) *η is an isomorphism and $G = (G, \eta^{-1})$ is a ∂-functor.*

(2) *Both ε and δ are homomorphisms of ∂-functors.*

Proof. We divide the proof into several steps. Note first that G is additive.

Claim 1: η is an isomorphism.

Proof. Let $M \in \mathrm{Ob}(\mathcal{H})$ and $X \in \mathrm{Ob}(\mathcal{K})$. For any $h \in \mathcal{H}(FT^{-1}X, M)$, since we have a commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{T\varepsilon_{T^{-1}X}} & TGFT^{-1}X & \xrightarrow{TGh} & TGM \\
\varepsilon_X \downarrow & & \downarrow \varepsilon_{TGFT^{-1}X} & & \downarrow \varepsilon_{TGM} \\
GFX & \xrightarrow{GFT\varepsilon_{T^{-1}X}} & GFTGFT^{-1}X & \xrightarrow{GFTGh} & GFTGM \\
G\theta_{T^{-1}X} \downarrow & & \downarrow G\theta_{GFT^{-1}X} & & \downarrow G\theta_{GM} \\
GTFT^{-1}X & \xrightarrow{GTF\varepsilon_{T^{-1}X}} & GTFGFT^{-1}X & \xrightarrow{GTFGh} & GTFGM \\
 & & \downarrow GT\delta_{FT^{-1}X} & & \downarrow GT\delta_M \\
 & & GTFT^{-1}X & \xrightarrow{GTh} & GTM,
\end{array}
$$

we have

$$
\begin{aligned}
G(T(h \circ T^{-1}\theta_{T^{-1}X})) \circ \varepsilon_X &= GTh \circ G\theta_{T^{-1}X} \circ \varepsilon_X \\
&= GTh \circ \mathrm{id}_{GTFT^{-1}X} \circ G\theta_{T^{-1}X} \circ \varepsilon_X \\
&= GTh \circ (GT\delta_{FT^{-1}X} \circ GTF\varepsilon_{T^{-1}X}) \circ G\theta_{T^{-1}X} \circ \varepsilon_X \\
&= GT\delta_M \circ G\theta_{GM} \circ \varepsilon_{TGM} \circ TGh \circ T\varepsilon_{T^{-1}X} \\
&= \eta_M \circ TGh \circ T\varepsilon_{T^{-1}X} \\
&= \eta_M \circ T(Gh \circ \varepsilon_{T^{-1}X}).
\end{aligned}
$$

Thus the following diagram commutes

$$\begin{array}{ccccc} \mathcal{H}(FT^{-1}X, M) & \tilde{\to} & \mathcal{K}(T^{-1}X, GM) & \tilde{\to} & \mathcal{K}(X, TGM) \\ {\scriptstyle \mathcal{H}(T^{-1}\theta_{T^{-1}X},\, \mathrm{id}_M)} \downarrow & & & & \downarrow {\scriptstyle \mathcal{K}(\mathrm{id}_X,\, \eta_M)} \\ \mathcal{H}(T^{-1}FX, M) & \tilde{\to} & \mathcal{K}(FX, TM) & \tilde{\to} & \mathcal{K}(X, GTM)\,. \end{array}$$

Since $\mathcal{H}(T^{-1}\theta_{T^{-1}X}, \mathrm{id}_M)$ is an isomorphism, so is $\mathcal{K}(\mathrm{id}_X, \eta_M)$. It follows by Yoneda lemma that η_M is an isomorphism.

Claim 2: $G = (G, \eta^{-1}) : \mathcal{H} \to \mathcal{K}$ is a ∂-functor.

Proof. Let (L, M, N, a, b, c) be a triangle in $\mathcal{H}$. Since by (TR1) we have a triangle in $\mathcal{K}$ of the form (GL, GM, Z, Ga, v, w), we have a triangle in $\mathcal{H}$

$$(FGL, FGM, FZ, FGa, Fv, \theta_{GL} \circ Fw).$$

Thus by (TR3) there exists $h \in \mathcal{H}(FZ, N)$ which makes the following diagram commute

$$\begin{array}{ccccccc} FGL & \xrightarrow{FGa} & FGM & \xrightarrow{Fv} & FZ & \xrightarrow{\theta_{GL} \circ Fw} & TFGL \\ \delta_L \downarrow & & \downarrow \delta_M & & \downarrow h & & \downarrow T\delta_L \\ L & \xrightarrow{a} & M & \xrightarrow{b} & N & \xrightarrow{c} & TL\,. \end{array}$$

Then we have

$$\begin{aligned} Gh \circ \varepsilon_Z \circ v &= Gh \circ GFv \circ \varepsilon_{GM} \\ &= G(h \circ Fv) \circ \varepsilon_{GM} \\ &= G(b \circ \delta_M) \circ \varepsilon_{GM} \\ &= Gb \circ G\delta_M \circ \varepsilon_{GM} \\ &= Gb \circ (G\delta \circ \varepsilon_G)_M \\ &= Gb \circ (\mathrm{id}_G)_M \\ &= Gb, \end{aligned}$$

$$\begin{aligned} Gc \circ Gh \circ \varepsilon_Z &= G(c \circ h) \circ \varepsilon_Z \\ &= G(T\delta_L \circ \theta_{GL} \circ Fw) \circ \varepsilon_Z \\ &= GT\delta_L \circ G\theta_{GL} \circ GFw \circ \varepsilon_Z \\ &= GT\delta_L \circ G\theta_{GL} \circ \varepsilon_{TGL} \circ w \end{aligned}$$

$$= \eta_L \circ w.$$

Thus the following diagram commutes

$$\begin{array}{ccccccc} GL & \xrightarrow{Ga} & GM & \xrightarrow{v} & Z & \xrightarrow{w} & TGL \\ \| & & \| & & \downarrow Gh \circ \varepsilon_Z & & \| \\ GH & \xrightarrow{Ga} & GM & \xrightarrow{Gb} & GN & \xrightarrow{\eta_L^{-1} \circ Gc} & TGL. \end{array}$$

We claim that $Gh \circ \varepsilon_Z$ is an isomorphism. Let $X \in \mathrm{Ob}(\mathcal{H})$. We have the following commutative diagram with the top and the bottom rows exact

$$\begin{array}{ccccccccc} \mathcal{H}(X, GL) & \to & \mathcal{H}(X, GM) & \to & \mathcal{H}(X, GN) & \to & \mathcal{H}(X, TGL) & \to & \mathcal{H}(X, TGM) \\ \| & & \| & & \downarrow \mathcal{H}(X, Gh \circ \varepsilon_Z) & & \| & & \| \\ \mathcal{H}(X, GL) & \to & \mathcal{H}(X, GM) & \to & \mathcal{H}(X, GN) & \to & \mathcal{H}(X, TGL) & \to & \mathcal{H}(X, TGM) \\ \| & & \| & & \| & & \downarrow \mathcal{H}(X, \eta_L) & & \downarrow \mathcal{H}(X, \eta_M) \\ \mathcal{H}(X, GL) & \to & \mathcal{H}(X, GM) & \to & \mathcal{H}(X, GN) & \to & \mathcal{H}(X, GTL) & \to & \mathcal{H}(X, GTM) \\ \phi_{X, L} \downarrow & & \downarrow \phi_{X, M} & & \downarrow \phi_{X, N} & & \downarrow \phi_{X, TL} & & \downarrow \phi_{X, TM} \\ \mathcal{H}(FX, L) & \to & \mathcal{H}(FX, M) & \to & \mathcal{H}(FX, N) & \to & \mathcal{H}(FX, TL) & \to & \mathcal{H}(FX, TM), \end{array}$$

where $\phi : \mathcal{H}(-, G-) \xrightarrow{\sim} \mathcal{H}(F-, -)$ is an isomorphism of bifunctors. Since $\mathcal{H}(X, \eta_L)$, $\mathcal{H}(X, \eta_M)$ are isomorphisms, the second row is exact. Thus by five-lemma $\mathcal{H}(X, Gh \circ \varepsilon_Z)$ is an isomorphism, so is $Gh \circ \varepsilon_Z$ by Yoneda lemma.

Claim 3: $\varepsilon : (\mathbf{1}_{\mathcal{H}}, \mathrm{id}) \to (GF, \eta_F^{-1} \circ G\theta)$is a homomorphism of ∂-functors.

Proof. By Proposition 7.10(4), $GF = (GF, \eta_F^{-1} \circ G\theta)$ is a ∂-functor. We have

$$\begin{aligned} \eta_F \circ T\varepsilon &= GT\delta_F \circ G\theta_{GF} \circ \varepsilon_{TGF} \circ T\varepsilon \\ &= GT\delta_F \circ G\theta_{GF} \circ GFT\varepsilon \circ \varepsilon_T \\ &= G(T\delta_F \circ \theta_{GF} \circ FT\varepsilon) \circ \varepsilon_T \\ &= G(T\delta_F \circ TF\varepsilon \circ \theta) \circ \varepsilon_T \\ &= GT(\delta_F \circ F\varepsilon) \circ G\theta \circ \varepsilon_T \\ &= GT(\mathrm{id}_F) \circ G\theta \circ \varepsilon_T \\ &= G\theta \circ \varepsilon_T. \end{aligned}$$

Thus $T\varepsilon = (\eta_F^{-1} \circ G\theta) \circ \varepsilon_T$.

Claim 4: $\delta : (FG,\ \theta_G \circ F\eta^{-1}) \to (\mathbf{1}_{\mathcal{H}},\ \mathrm{id})$ is a homomorphism of ∂-functors.

Proof. By Proposition 7.10(4), $FG = (FG,\ \theta_G \circ F\eta^{-1})$ is a ∂-functor. We have

$$\begin{aligned}
\delta_T \circ F\eta &= \delta_T \circ FGT\delta \circ FG\theta_G \circ F\varepsilon_{TG} \\
&= T\delta \circ \delta_{TFG} \circ FG\theta_G \circ F\varepsilon_{TG} \\
&= T\delta \circ (\delta_{TF} \circ FG\theta)_G \circ F\varepsilon_{TG} \\
&= T\delta \circ (\theta \circ \delta_{FT})_G \circ F\varepsilon_{TG} \\
&= T\delta \circ \theta_G \circ (\delta_F \circ F\varepsilon)_{TG} \\
&= T\delta \circ \theta_G \circ (\mathrm{id}_F)_{TG} \\
&= T\delta \circ \theta_G .
\end{aligned}$$

Thus $\delta_T = T\delta \circ (\theta_G \circ F\eta^{-1})$.

Proposition 12.2 (Dual of Proposition 12.1). *Let $F = (F,\ \theta) : \mathcal{K} \to \mathcal{H}$ be a ∂-functor. Assume F has a right adjoint $G : \mathcal{H} \to \mathcal{K}$. Let $\varepsilon : \mathbf{1}_{\mathcal{H}} \to FG$, $\delta : GF \to \mathbf{1}_{\mathcal{K}}$ be the unit and the counit, respectively, and put $\eta = \delta_{TG} \circ G\theta_G \circ GT\varepsilon : TG \to GT$. Then the following hold.*

(1) *η is an isomorphism and $G = (G,\ \eta^{-1})$ is a ∂-functor.*

(2) *Both ε and δ are homomorphisms of ∂-functors.*

Proposition 12.3. *Let $F : \mathcal{K} \to \mathcal{H}$ be a ∂-functor with a fully faithful right adjoint $G : \mathcal{H} \to \mathcal{K}$ and let $\varepsilon : \mathbf{1}_{\mathcal{K}} \to GF$, $\delta : FG \to \mathbf{1}_{\mathcal{H}}$ be the unit and the counit, respectively. Then the following hold.*

(1) *$\delta : FG \to \mathbf{1}_{\mathcal{H}}$ is an isomorphism.*

(2) Ker F *is an épaisse subcategory of* $\mathcal{K}$.

(3) $\varepsilon_X \in \Phi(\mathrm{Ker}\,F)$ *for all* $X \in \mathrm{Ob}(\mathcal{K})$.

(4) *The induced functor $\overline{F} : \mathcal{K}/\mathrm{Ker}\,F \to \mathcal{H}$ is an equivalence of triangulated categories.*

Proof. (1) Well-known.

(2) By Proposition 7.12(1).

(3) Since $\delta_F \circ F\varepsilon = \mathrm{id}_F$, by the part (1) $F\varepsilon$ is an isomorphism. Thus, for any $X \in \mathrm{Ob}(\mathcal{K})$, $F\varepsilon_X$ is an isomorphism and by Proposition 7.12(2) we have $\varepsilon_X \in \Phi(\mathrm{Ker}\,F)$.

(4) Let $Q : \mathcal{K} \to \mathcal{K}/\mathrm{Ker}\,F$ be the canonical functor. Then by Proposition 9.10 there exists a unique ∂-functor $\overline{F} : \mathcal{K}/\mathrm{Ker}\,F \to \mathcal{H}$ such that $F = \overline{F}Q$. We claim that $QG : \mathcal{H} \to \mathcal{K}/\mathrm{Ker}\,F$ is a quasi-inverse of $\overline{F}$. By the part (3) and Corollary 8.9 $Q\varepsilon : Q \to QGF = QG\overline{F}Q$ is an isomorphism. Thus by Proposition 8.11 there exists an isomorphism $\tau : \mathbf{1}_{\mathcal{K}/\mathrm{Ker}\,F} \to QG\overline{F}$ such that $Q\varepsilon = \tau_Q$.

Definition 12.1. A ∂-functor $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ is called a localization if it has a fully faithful right adjoint, i.e., F has a right adjoint $G : \mathcal{H} \to \mathcal{K}$ such that the counit $\delta : FG \to \mathbf{1}_{\mathcal{H}}$ is an isomorphism. If this is the case, $GF : \mathcal{K} \to \mathcal{K}$ is called a localization functor.

Proposition 12.4. *Assume $\mathcal{A}$ has enough injectives. Then the following hold.*

(1) *The canonical functor $Q : K^{+}(\mathcal{A}) \to D^{+}(\mathcal{A})$ is a localization.*

(2) *The canonical functor $Q : K^{+,\,\mathrm{b}}(\mathcal{A}) \to D^{+,\,\mathrm{b}}(\mathcal{A})$ is a localization.*

(3) *The canonical functor $Q : K^{+}(\mathcal{A})_{\mathrm{fid}} \to D^{+}(\mathcal{A})_{\mathrm{fid}}$ is a localization.*

Proof. (1) Let $J : K^{+}(\mathcal{I}) \to K^{+}(\mathcal{A})$ be the inclusion. Then by Proposition 10.13 $QJ : K^{+}(\mathcal{I}) \to D^{+}(\mathcal{A})$ is an equivalence. Let $P : D^{+}(\mathcal{A}) \to K^{+}(\mathcal{I})$ be a quasi-inverse of QJ. It follows by Proposition 10.12 that JP is a right adjoint of Q. Since P is fully faithful, so is JP.

(2) Replace $K^{+}(\mathcal{I})$ with $K^{+,\,\mathrm{b}}(\mathcal{I})$ in the proof of (1).

(3) Let $J : K^{\mathrm{b}}(\mathcal{I}) \to K^{+}(\mathcal{A})_{\mathrm{fid}}$ be the inclusion. Then by Proposition 11.13 $QJ : K^{\mathrm{b}}(\mathcal{I}) \to D^{+}(\mathcal{A})_{\mathrm{fid}}$ is an equivalence. Let $P : D^{+}(\mathcal{A})_{\mathrm{fid}} \to K^{\mathrm{b}}(\mathcal{I})$ be a quasi-inverse of QJ. Then, as in the part (1), JP is a fully faithful right adjoint of Q.

Proposition 12.5. *Let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$ with enough $\mathcal{A}$-injectives. Then the canonical functor $Q : K^{+}_{\mathcal{A}'}(\mathcal{A}) \to D^{+}_{\mathcal{A}'}(\mathcal{A})$ is a localization.*

Proof. Let $J : K^{+}(\mathcal{A}' \cap \mathcal{I}) \to K^{+}_{\mathcal{A}'}(\mathcal{A})$ be the inclusion. Then by Proposition 10.19 $QJ : K^{+}(\mathcal{A}' \cap \mathcal{I}) \to D^{+}_{\mathcal{A}'}(\mathcal{A})$ is an equivalence. Let $P : D^{+}_{\mathcal{A}'}(\mathcal{A}) \to K^{+}(\mathcal{A}' \cap \mathcal{I})$ be a quasi-inverse of QJ. Then, as in Proposition 12.4, JP is a fully faithful right adjoint of Q.

Proposition 12.6 (Dual of Proposition 12.3). *Let $F : \mathcal{K} \to \mathcal{H}$ be a ∂-functor with a fully faithful left adjoint $G : \mathcal{H} \to \mathcal{K}$ and let $\varepsilon : \mathbf{1}_{\mathcal{H}} \to FG$, $\delta : GF \to \mathbf{1}_{\mathcal{K}}$ be the unit and the counit, respectively. Then the following hold.*

(1) *$\varepsilon : \mathbf{1}_{\mathcal{H}} \to FG$is an isomorphism.*

(2) $\mathrm{Ker}\,F$ *is an épaisse subcategory of* $\mathcal{K}$.

(3) $\delta_X \in \Phi(\mathrm{Ker}\,F)$ *for all* $X \in \mathrm{Ob}(\mathcal{K})$.

(4) *The induced functor $\overline{F} : \mathcal{K}/\mathrm{Ker}\,F \to \mathcal{H}$ is an equivalence of triangulated categories.*

Definition 12.2. A ∂-functor $F = (F, \theta) : \mathcal{K} \to \mathcal{H}$ is called a colocalization if it has a fully faithful left adjoint, i.e., F has a left adjoint $G : \mathcal{H} \to \mathcal{K}$ such that the unit $\varepsilon : \mathbf{1}_{\mathcal{H}} \to FG$ is an isomorphism. If this is the case, $GF : \mathcal{K} \to \mathcal{K}$ is called a colocalization functor.

Proposition 12.7 (Dual of Proposition 12.4). *Assume $\mathcal{A}$ has enough projectives. Then the following hold.*

(1) *The canonical functor $Q : K^{-}(\mathcal{A}) \to D^{-}(\mathcal{A})$ is a colocalization.*

(2) *The canonical functor* $Q : K^{-,\mathrm{b}}(\mathscr{A}) \to D^{-,\mathrm{b}}(\mathscr{A})$ *is a colocalization*.
(3) *The canonical functor* $Q : K^{-}(\mathscr{A})_{\mathrm{fid}} \to D^{-}(\mathscr{A})_{\mathrm{fpd}}$ *is a colocalization*.

Proposition 12.8 (Dual of Proposition 12.5). *Let* $\mathscr{A}'$ *be a thick subcategory of* $\mathscr{A}$ *with enough* $\mathscr{A}$*-projectives*. *Then the canonical functor* $Q : K^{-}_{\mathscr{A}'}(\mathscr{A}) \to D^{-}_{\mathscr{A}'}(\mathscr{A})$ *is a colocalization.*

Proposition 12.9. *Let* $\mathscr{B}$ *be another abelian category and* $F : \mathscr{A} \to \mathscr{B}$ *an additive functor. Assume F has a fully faithful right (resp. left) adjoint. Then the extended* ∂*-functor* $F : K(\mathscr{A}) \to K(\mathscr{B})$ *is a localization (resp. colocalization).*

Proof. By Proposition 3.10.

Corollary 12.10. *Let* $\varphi : A \to B$ *be a ring epimorphism. Then the following hold*.
(1) $B \otimes_A - : K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, B)$ *is a localization*.
(2) $\mathrm{Hom}_A({}_AB_B, -) : K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, B)$ *is a colocalization*.

Proof. Let $U : \mathrm{Mod}\, B \to \mathrm{Mod}\, A$ be the canonical functor induced by $\varphi : A \to B$. Then U is a right adjoint of $B \otimes_A -$ and a left adjoint of $\mathrm{Hom}_A({}_AB_B, -)$. Also, by assumption, U is fully faithful.

Corollary 12.11. *Let A be a ring and* $e \in A$ *an idempotent*. *Then*

$$eA \otimes_A - \cong \mathrm{Hom}_A(Ae, -) : K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, eAe)$$

is a bilocalization, i.e., both a localization and a colocalization, simultaneously.

Proof. It is obvious that $eA \otimes_A - \cong \mathrm{Hom}_A(Ae, -)$. Also, $eA \otimes_A -$ has a fully faithful right adjoint $\mathrm{Hom}_{eAe}({}_{eAe}eA_A, -)$ and $\mathrm{Hom}_A(Ae, -)$ has a fully faithful left adjoint $Ae \otimes_{eAe} -$.

Definition 12.3. Let $\mathscr{C}$ be a category and Λ a small (connected) category. We denote by $\mathscr{C}^{\Lambda}$ the category of functors from Λ to $\mathscr{C}$: an object $F \in \mathrm{Ob}(\mathscr{C}^{\Lambda})$ is a pair $(\{F_\lambda\}, \{F_\alpha\})$ of a family of objects $\{F_\lambda\}_{\lambda \in \mathrm{Ob}(\Lambda)}$ in $\mathscr{C}$ and a family of morphisms $\{F_\alpha\}_{\alpha \in \mathrm{Mor}(\Lambda)}$ in $\mathscr{C}$, where $F_\alpha \in \mathscr{C}(F_\lambda, F_\mu)$ for $\alpha \in \Lambda(\lambda, \mu)$; and a morphism $h : (\{F_\alpha\}, \{F_\lambda\}) \to (\{G_\alpha\}, \{G_\lambda\})$ is a family of morphisms $\{h_\lambda\}_{\lambda \in \mathrm{Ob}(\Lambda)}$ in $\mathscr{C}$ such that $G_\alpha \circ h_\lambda = h_\mu \circ F_\alpha$ for all morphisms $\alpha \in \Lambda(\lambda, \mu)$. The constant functor

$$P : \mathscr{C} \to \mathscr{C}^{\Lambda}$$

associates with each $X \in \mathrm{Ob}(\mathscr{C})$ a pair $(\{F_\lambda\}, \{F_\alpha\})$ such that $F_\lambda = X$ for all $\lambda \in \mathrm{Ob}(\Lambda)$ and F_α

$= \mathrm{id}_X$ for all $\alpha \in \mathrm{Mor}(\Lambda)$.

Definition 12.4. Let $\mathscr{C}$ be a category and Λ a small (connected) category. A limit of $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$, denoted by $\varprojlim F$, is defined as a terminal object in the following category: an object is a morphism in $\mathscr{C}^\Lambda$ of the form $f : PX \to F$ with $X \in \mathrm{Ob}(\mathscr{C})$, i.e., a pair $(X, \{f_\lambda\})$ of $X \in \mathrm{Ob}(\mathscr{C})$ and a family of morphisms $f_\lambda \in \mathscr{C}(X, F_\lambda)$ with $f_\mu = F_\alpha \circ f_\lambda$ for all morphisms $\alpha \in \Lambda(\lambda, \mu)$; a morphism $h : (X, \{f_\lambda\}) \to (Y, \{g_\lambda\})$ is a morphism $h \in \mathscr{C}(X, Y)$ with $f_\lambda = g_\lambda \circ h$ for all $\lambda \in \mathrm{Ob}(\Lambda)$.

Remark 12.1. Assume every $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a limit $\varprojlim F = (\varprojlim F, \{p_\lambda\})$. Then $\varprojlim : \mathscr{C}^\Lambda \to \mathscr{C}$ is a functor and is a right adjoint of the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$. Furthermore, the morphisms $p_F = \{p_\lambda\} : P(\varprojlim F) \to F$ give rise to the counit $P \circ \varprojlim \to \mathbf{1}_{\mathscr{C}^\Lambda}$. In particular, if $\mathscr{C}$ is abelian, then $\varprojlim$ is left exact. Conversely, assume the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$ has a right adjoint $\varprojlim : \mathscr{C}^\Lambda \to \mathscr{C}$ and let $p : P \circ \varprojlim \to \mathbf{1}_{\mathscr{C}^\Lambda}$ be the counit. Then every $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a limit $\varprojlim F = (\varprojlim F,\ p_F)$.

Definition 12.5. We denote by $\mathbb{N}$ the totally ordered set of non-negative integers. In this case, a functor $\mathbb{N}^{\mathrm{op}} \to \mathscr{C}$ is given by a sequence of objects and morphisms in $\mathscr{C}$

$$\cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \to \cdots \to X_0$$

and its limit is denoted by $\varprojlim X_n$. In case $\mathscr{C}$ has countable direct products, there exists a unique morphism in $\mathscr{C}$

$$\mathrm{shift} : \prod X_n \to \prod X_n$$

such that $p_m \circ (\mathrm{shift}) = f_{m+1} \circ p_{m+1}$ for all $m \in \mathbb{N}$, where the $p_m : \prod X_n \to X_m$ are projections.

Lemma 12.12. *Each complex* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *defines a sequence of truncated complexes and canonical homomorphisms*

$$\cdots \to \sigma'_{\geq -(n+1)}(X^\bullet) \to \sigma'_{\geq -n}(X^\bullet) \to \cdots \to \sigma'_{\geq 0}(X^\bullet)$$

such that $X^\bullet \xrightarrow{\sim} \varprojlim \sigma'_{\geq -n}(X^\bullet)$ *canonically.*

Proof. Straightforward.

Lemma 12.13. *Assume* $\mathscr{A}$ *satisfies the condition Ab3*. Then for any sequence*

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_0$$

of objects and morphisms in $\mathscr{A}$ *we have an exact sequence in* $\mathscr{A}$

$$0 \longrightarrow \varprojlim X_n \longrightarrow \prod X_n \xrightarrow{1-\text{shift}} \prod X_n.$$

Proof. Straightforward.

Definition 12.6. Let $\mathscr{K}$ be a triangulated category with countable direct products. Then for a sequence of objects and morphisms

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_0,$$

its homotopy limit, denoted by $\mathrm{h}\varprojlim X_n$, is defined by a triangle

$$T^{-1}(\prod X_n) \longrightarrow \mathrm{h}\varprojlim X_n \longrightarrow \prod X_n \xrightarrow{1-\text{shift}} \prod X_n.$$

Definition 12.7. For * = – or nothing, we denote by $K^*(\mathscr{I})_{\mathrm{L}}$ the full subcategory of $K^*(\mathscr{A})$ consisting of $\mathscr{U}$-local complexes $I^\bullet \in \mathrm{Ob}(K^*(\mathscr{I}))$.

Remark 12.2. It follows by Lemma 4.4 that $K^{\mathrm{b}}(\mathscr{I}) \subset K^{+}(\mathscr{I}) \subset K(\mathscr{I})_{\mathrm{L}}$.

Lemma 12.14. $K(\mathscr{I})_{\mathrm{L}}$ *is a full triangulated subcategory of* $K(\mathscr{A})$ *closed under direct products and* $\mathscr{U} \cap K(\mathscr{I})_{\mathrm{L}} = \{0\}$.

Proof. The first assertion is obvious. For any $I^\bullet \in \mathrm{Ob}(\mathscr{U} \cap K(\mathscr{I})_{\mathrm{L}})$, since $K(\mathscr{A})(I^\bullet, I^\bullet) = 0$, by Proposition 3.5 $I^\bullet = 0$ in $K(\mathscr{A})$.

Proposition 12.15. *Assume* $\mathscr{A}$ *has enough injectives and satisfies the condition Ab4**. *Then the following hold.*

(1) *For any* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$.

(2) *For any* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A})_{\mathrm{fid}})$ *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(K^{-}(\mathscr{I})_{\mathrm{L}})$.

Proof. (1) Let $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ and put $X^\bullet_m = \sigma'_{\geq -m}(X^\bullet)$ for $m \in \mathbb{N}$.

Claim 1: There exists a quasi-isomorphism $\phi : X^\bullet \to \operatorname{h}\lim\limits_{\leftarrow} X^\bullet_m$.

Proof. By Lemmas 12.12 and 12.13 we have an exact sequence in $C(\mathscr{A})$

$$0 \longrightarrow X^\bullet \longrightarrow \prod X^\bullet_m \xrightarrow{1-\text{shift}} \prod X^\bullet_m .$$

Thus by Proposition 6.5(1) we have a commutative diagram in $K(\mathscr{A})$

$$\begin{array}{ccccc} X^\bullet & \longrightarrow & \prod X^\bullet_m & \xrightarrow{1-\text{shift}} & \prod X^\bullet_m \\ {\scriptstyle\phi}\downarrow & & \| & & \| \\ \operatorname{h}\lim\limits_{\leftarrow} X^\bullet_m & \longrightarrow & \prod X^\bullet_m & \xrightarrow{1-\text{shift}} & \prod X^\bullet_m . \end{array}$$

We claim that ϕ is a quasi-isomorphism. It suffices to show that $H^n(\phi)$ is an isomorphis for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. We have a sequence of objects and morphisms in $\mathscr{A}$

$$\cdots \to H^n(X^\bullet_{m+1}) \to H^n(X^\bullet_m) \to \cdots \to H^n(X^\bullet_0)$$

such that $H^n(X^\bullet) = H^n(X^\bullet_m)$ for $-m \le n$ and $H^n(X^\bullet_m) = 0$ for $-m > n$. Thus we have an exact sequence

$$0 \longrightarrow H^n(X^\bullet) \longrightarrow \prod H^n(X^\bullet_m) \xrightarrow{1-\text{shift}} \prod H^n(X^\bullet_m) \longrightarrow 0.$$

Note that $H^n(\prod X^\bullet_m) \cong \prod H^n(X^\bullet_m)$. Thus, $H^n(1 - \text{shift})$ is epic and we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n(X^\bullet) & \longrightarrow & H^n(\prod X^\bullet_m) & \longrightarrow & H^n(\prod X^\bullet_m) & \longrightarrow & 0 \\ & & {\scriptstyle H^n(\phi)}\downarrow & & \| & & \| & & \\ 0 & \to & H^n(\operatorname{h}\lim\limits_{\leftarrow} X^\bullet_m) & \to & H^n(\prod X^\bullet_m) & \longrightarrow & H^n(\prod X^\bullet_m) & \longrightarrow & 0 . \end{array}$$

Claim 2: For each $m \in \mathbb{N}$ there exists a quasi-isomorphism $\psi_m : X^\bullet_m \to I^\bullet_m$ with $\in$ $\mathrm{Ob}(K^+(\mathscr{I}))$.

Proof. By Proposition 4.7.

Claim 3: For each $m \in \mathbb{N}$ there exists a morphism $I^\bullet_{m+1} \to I^\bullet_m$ in $K(\mathscr{A})$ which makes the following diagram commute

$$\begin{array}{ccc} X^\bullet_{m+1} & \to & X^\bullet_m \\ \downarrow & & \downarrow \\ I^\bullet_{m+1} & \to & I^\bullet_m . \end{array}$$

Proof. By Lemma 6.15(1).

Claim 4: $\mathrm{h}\varprojlim \psi_m : \mathrm{h}\varprojlim X^\bullet_m \to \mathrm{h}\varprojlim I^\bullet_m$ is a quasi-isomorphism.

Proof. By Claim 3 we have a homomorphism of triangles in $K(\mathscr{A})$

$$\begin{array}{ccccccc} T^{-1}(\prod X^\bullet_m) & \longrightarrow & \mathrm{h}\varprojlim X^\bullet_m & \longrightarrow & \prod X^\bullet_m & \xrightarrow{1-\mathrm{shift}} & \prod X^\bullet_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{-1}(\prod I^\bullet_m) & \longrightarrow & \mathrm{h}\varprojlim I^\bullet_m & \longrightarrow & \prod I^\bullet_m & \xrightarrow{1-\mathrm{shift}} & \prod I^\bullet_m . \end{array}$$

Since $H^n(\prod \psi_m) \cong \prod H^n(\psi_m)$ is an isomorphism for all $n \in \mathbb{Z}$, $\prod \psi_m$ is a quasi-isomorphism, so is $\mathrm{h}\varprojlim \psi_m : \mathrm{h}\varprojlim X^\bullet_m \to \mathrm{h}\varprojlim I^\bullet_m$.

Claim 5: $\mathrm{h}\varprojlim I^\bullet_m$ is $\mathscr{U}$-local.

Proof. By Lemma 4.4 every $I^\bullet_m$ is $\mathscr{U}$-local. Thus by Proposition 9.12 $\prod I^\bullet_m$ is $\mathscr{U}$-local, so is $\mathrm{h}\varprojlim I^\bullet_m$.

(2) Let $n \in \mathbb{Z}$ and assume $\mathrm{Ext}^i(-, X^\bullet)$ vanishes on $\mathscr{A}$ for $i > n$. By the part (1) there exists a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$. Let $i > n$ and $j : Z^i(I^\bullet) \to I^i$ the inclusion. Since by Proposition 9.13(2) we have

$$\begin{aligned} K(\mathscr{A})(T^{-i}(Z^i(I^\bullet)), I^\bullet) &\cong K(\mathscr{A})(Z^i(I^\bullet), T^i(I^\bullet)) \\ &\cong D(\mathscr{A})(Z^i(I^\bullet), T^i(I^\bullet)) \\ &\cong \mathrm{Ext}^i(Z^i(I^\bullet), I^\bullet) \\ &= 0, \end{aligned}$$

there exists $f : Z^i(I^\bullet) \to I^{i-1}$ such that $j = d_I^{i-1} \circ f$. Thus $B^i(I^\bullet) = Z^i(I^\bullet)$ and the canonical epimorphism $I^{i-1} \to B^i(I^\bullet)$ splits. Consequently, $H^i(I^\bullet) = 0$ and $Z^{i-1}(I^\bullet) \in \mathscr{I}$ for all $i > n$. Thus $\sigma_n(I^\bullet) \in \mathrm{Ob}(K^-(\mathscr{I}))$ and by Lemma 10.7 the canonical monomorphism $\sigma_n(I^\bullet) \to I^\bullet$ is a quasi-isomorphism.

Claim 6: $\sigma_n(I^\bullet) \in \mathrm{Ob}(K^-(\mathscr{I})_{\mathrm{L}})$ and we have a quasi-isomorphism $X^\bullet \to \sigma_n(I^\bullet)$.

Proof. Let $Z^\bullet \in \mathrm{Ob}(\mathcal{U})$ and $u \in K(\mathcal{A})(Z^\bullet, \sigma_{\geq n}(I^\bullet))$. We claim that $u = 0$ in $K(\mathcal{A})$. Let $j : Z^n(I^\bullet) \to I^n$ be the inclusion and $\hat{j} : \sigma_{\geq n}(I^\bullet) \to I^\bullet$ the canonical monomorphism. Then, since $Z^n(I^\bullet) \in \mathcal{I}$, there exists $g : I^n \to Z^n(I^\bullet)$ such that $gj = \mathrm{id}_{Z^n(I^\bullet)}$. Also, since $I^\bullet \in \mathrm{Ob}(K(\mathcal{I})_{\mathrm{L}})$, there exists $h : \hat{j}\,u \simeq 0$. Define $h' \in \mathcal{A}^{\mathbb{Z}}(T(\sigma_{\geq n}(I^\bullet)), Z^\bullet)$ as follows:

$$h'^i = \begin{cases} 0 & (i > n) \\ g \circ h^n & (i = n). \\ h^i & (i < n) \end{cases}$$

Then it is easy to see that $h' : u \simeq 0$. Thus $\sigma_{\geq n}(I^\bullet) \in \mathrm{Ob}(K^-(\mathcal{I})_{\mathrm{L}})$ and by Proposition 9.13(1) $\hat{j} : \sigma_{\geq n}(I^\bullet) \to I^\bullet$ is an isomorphism in $K(\mathcal{A})$. Hence we get a quasi-isomorphism $\hat{j}^{-1} \circ s : X^\bullet \to \sigma_{\geq n}(I^\bullet)$.

Proposition 12.16. *Assume $\mathcal{A}$ has enough injectives and satisfies the condition Ab4*[*]. *Then the following hold.*

(1) *The canonical functor $K(\mathcal{A}) \to D(\mathcal{A})$ induces equivalences $K(\mathcal{I})_{\mathrm{L}} \xrightarrow{\sim} D(\mathcal{A})$ and $K^-(\mathcal{I})_{\mathrm{L}} \xrightarrow{\sim} D(\mathcal{A})_{\mathrm{fid}}$.*

(2) *The canonical functors $K(\mathcal{A}) \to D(\mathcal{A})$ and $K(\mathcal{A})_{\mathrm{fid}} \to D(\mathcal{A})_{\mathrm{fid}}$ are localizations.*

(3) *The canonical functor $D(\mathcal{A})_{\mathrm{fid}} \to D(\mathcal{A})$ preserves direct products.*

Proof. (1) By Propositions 9.13(2) and 12.15.

(2) Let $Q : K(\mathcal{A}) \to D(\mathcal{A})$ be the canonical functor and $J : K(\mathcal{I})_{\mathrm{L}} \to K(\mathcal{A})$ the inclusion. Then by the part (1) $QJ : K(\mathcal{I})_{\mathrm{L}} \to D(\mathcal{A})$ is an equivalence. Let $P : D(\mathcal{A}) \to K(\mathcal{I})_{\mathrm{L}}$ be a quasi-inverse of QJ. Then by Proposition 9.13(2) JP is a right adjoint of Q. Since P is fully faithful, so is JP. Similarly, $K(\mathcal{A})_{\mathrm{fid}} \to D(\mathcal{A})_{\mathrm{fid}}$ is a localization.

(3) Let $\{X^\bullet_\lambda\}_{\lambda \in \Lambda}$ be a family of objects of $D(\mathcal{A})_{\mathrm{fid}}$ which has a direct product $X^\bullet$ in $D(\mathcal{A})_{\mathrm{fid}}$. We claim $\prod X^\bullet_\lambda \in \mathrm{Ob}(D(\mathcal{A})_{\mathrm{fid}})$. By Proposition 12.15(2) we may assume the $X^\bullet_\lambda$ and $X^\bullet$ are objects of $K^-(\mathcal{I})_{\mathrm{L}}$. Take $a \in \mathbb{Z}$ such that $X^i = 0$ for $i > a$. Let $\lambda \in \Lambda$ and put $n = \max\{i \in \mathbb{Z} \mid H^i(X^\bullet_\lambda) \quad 0\}$. By Lemma 10.6 we have a quasi-isomorphism $t : \sigma_{\geq n}(X^\bullet_\lambda) \to X^\bullet_\lambda$. Also, there exists $v : T^{-n}(Z^n(X^\bullet_\lambda)) \to \sigma_{\geq n}(X^\bullet_\lambda)$ such that $H^n(v) : Z^n(X^\bullet_\lambda) \to H^n(X^\bullet_\lambda)$ is the canonical epimorphism. Thus $0 \quad Q(tv) \in D(\mathcal{A})(T^{-n}(Z^n(X^\bullet_\lambda)), X^\bullet_\lambda)$. Since $Z^n(X^\bullet_\lambda)$ admits an injective resolution

$$0 \to Z^n(X^\bullet_\lambda) \to X^n_\lambda \to X^{n+1}_\lambda \to \cdots,$$

$X^\bullet_\lambda \in \mathrm{Ob}(K^-(\mathcal{I})_{\mathrm{L}})$ implies $T^{-n}(Z^n(X^\bullet_\lambda)) \in \mathrm{Ob}(D(\mathcal{A})_{\mathrm{fid}})$. Hence $D(\mathcal{A})(T^{-n}(Z^n(X^\bullet_\lambda)), X^\bullet) \quad 0$ and $n \leq a$. Also, since $D(\mathcal{A})(T^{-(a+1)}(B'^a(X^\bullet_\lambda)), X^\bullet) = 0$, by Proposition 9.13(2) we have

$$K(\mathscr{A})(T^{-(a+1)}(B'^a(X^\bullet_\lambda)),\ X^\bullet_\lambda) \cong D(\mathscr{A})(T^{-(a+1)}(B'^a(X^\bullet_\lambda)),\ X^\bullet_\lambda)$$
$$= 0$$

and the canonical exact sequence $0 \to Z^a(X^\bullet_\lambda) \to X^a_\lambda \to B'^a(X^\bullet_\lambda) \to 0$ splits. Thus $\sigma_{\le a}(X^\bullet_\lambda) \in \mathrm{Ob}(K^-(\mathscr{I}))$ and $\sigma_{>a}(X^\bullet_\lambda) \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$. Then, since by Lemma 4.4 $K^{\mathrm{b}}(\mathscr{I}) \subset K^-(\mathscr{I})_{\mathrm{L}}$, and since by Proposition 11.1(3) we have a triangle of the form

$$(\sigma_{\le a}(X^\bullet_\lambda),\ X^\bullet_\lambda,\ \sigma_{>a}(X^\bullet_\lambda),\ \cdot,\ \cdot,\ \cdot),$$

it follows that $\sigma_{\le a}(X^\bullet_\lambda) \in \mathrm{Ob}(K^-(\mathscr{I})_{\mathrm{L}})$. Thus we have a quasi-isomorphism $\sigma_{\le a}(X^\bullet_\lambda) \to X^\bullet_\lambda$ with $\sigma_{\le a}(X^\bullet_\lambda) \in \mathrm{Ob}(K^-(\mathscr{I})_{\mathrm{L}})$. Consequently, we may assume $X^i_\lambda = 0$ for all $i > a$ and $\lambda \in \Lambda$. Then $\prod X^\bullet_\lambda \in \mathrm{Ob}(K^-(\mathscr{I})_{\mathrm{L}})$ and $\prod X^\bullet_\lambda \cong X^\bullet$.

Definition 12.8. Let $\mathscr{C}$ be a category and Λ a small (connected) category. A colimit of $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$, denoted by $\varinjlim F$, is defined as an initial object in the following category: an object is a morphism in $\mathscr{C}^\Lambda$ of the form $f : F \to PX$ with $X \in \mathrm{Ob}(\mathscr{C})$, i.e., a pair $(\{f_\lambda\}, X)$ of $X \in \mathrm{Ob}(\mathscr{C})$ and a family of morphisms $f_\lambda \in \mathscr{C}(X, F_\lambda)$ with $f_\mu \circ F_\alpha = f_\lambda$ for all $\alpha \in \Lambda(\lambda, \mu)$; a morphism $h : (\{f_\lambda\}, X) \to (\{g_\lambda\}, Y)$ is a morphism $h \in \mathscr{C}(X, Y)$ with $g_\lambda = h \circ g_\lambda$ for all $\lambda \in \mathrm{Ob}(\Lambda)$.

Remark 12.3. Assume every $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a colimit $\varinjlim F = (\{i_\lambda\}, \varinjlim F)$. Then $\varinjlim : \mathscr{C}^\Lambda \to \mathscr{C}$ is a functor and is a left adjoint of the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$. Furthermore, the morphisms $i_F = \{i_\lambda\} : P(\varinjlim F) \to F$ gives rise to the unit $\mathbf{1}_{\mathscr{C}^\Lambda} \to P \circ \varinjlim$. In particular, if $\mathscr{C}$ is abelian, then is $\varinjlim$ right exact. Conversely, assume the constant functor $P : \mathscr{C} \to \mathscr{C}^\Lambda$ has a left adjoint $\varinjlim : \mathscr{C}^\Lambda \to \mathscr{C}$ and let $i : \mathbf{1}_{\mathscr{C}^\Lambda} \to P \circ \varinjlim$ be the unit. Then every $F \in \mathrm{Ob}(\mathscr{C}^\Lambda)$ has a colimit $\varinjlim F = (i_F, \varinjlim F)$.

Definition 12.9. A functor $\mathbb{N} \to \mathscr{C}$ is given by a sequence of objects and morphisms in $\mathscr{C}$

$$X_0 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

and its colimit is denoted by $\varinjlim X_n$. In case $\mathscr{C}$ has countable direct sums, there exists a unique morphism in $\mathscr{C}$

$$\mathrm{shift} : \oplus X_n \to \oplus X_n$$

such that $(\mathrm{shift}) \circ i_m = i_{m+1} \circ f_m$ for all $m \in \mathbb{N}$, where the $i_m : X_m \to \oplus X_n$ are injections.

Lemma 12.17 (Dual of Lemma 12.12). *Each complex* $X^\bullet \in \mathrm{Ob}(C(\mathcal{A}))$ *defines a sequence of truncated complexes and canonical homomorphisms*

$$\sigma_0(X^\bullet) \to \cdots \to \sigma_n(X^\bullet) \to \sigma_{n+1}(X^\bullet) \to \cdots$$

such that $\varinjlim \sigma_n(X^\bullet) \tilde{\to} X^\bullet$ *canonically.*

Lemma 12.18 (Dual of Lemma 12.13). *Assume* $\mathcal{A}$ *satisfies the condition Ab3. Then for any sequence*

$$X_0 \to \cdots \to X_n \to X_{n+1} \to \cdots$$

of objects and morphisms in $\mathcal{A}$ *we have an exact sequence in* $\mathcal{A}$

$$\oplus X_n \xrightarrow{1-\text{shift}} \oplus X_n \longrightarrow \varinjlim X_n \longrightarrow 0.$$

Definition 12.10. Let $\mathcal{K}$ be a triangulated category with countable direct sums. Then for a sequence of objects and morphisms

$$X_0 \to \cdots \to X_n \to X_{n+1} \to \cdots ,$$

its homotopy colimit, denoted by $\mathrm{h}\varinjlim X_n$, is defined by a triangle

$$\oplus X_n \xrightarrow{1-\text{shift}} \oplus X_n \longrightarrow \mathrm{h}\varinjlim X_n \longrightarrow T(\oplus X_n).$$

Definition 12.11. For * = + or nothing, we denote by $K^*(\mathcal{P})_{\mathrm{L}}$ the full subcategory of $K^*(\mathcal{A})$ consisting of $\mathcal{U}$-colocal complexes $P^\bullet \in \mathrm{Ob}(K^*(\mathcal{P}))$.

Remark 12.4. It follows by Lemma 4.8 that $K^{\mathrm{b}}(\mathcal{P}) \subset K^-(\mathcal{P}) \subset K(\mathcal{P})_{\mathrm{L}}$.

Lemma 12.19 (Dual of Lemma 12.14). $K(\mathcal{P})_{\mathrm{L}}$ *is a full triangulated subcategory of* $K(\mathcal{A})$ *closed under direct sums and* $\mathcal{U} \cap K(\mathcal{P})_{\mathrm{L}} = \{0\}$.

Proposition 12.20 (Dual of Proposition 12.15). *Assume* $\mathcal{A}$ *has enough projectives and satisfies the condition Ab4. Then the following hold.*

(1) *For any* $X^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$ *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(K(\mathcal{P})_{\mathrm{L}})$.

(2) *For any* $X^\bullet \in \mathrm{Ob}(K(\mathcal{A})_{\mathrm{fpd}})$ *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in$

$\mathrm{Ob}(K^{+}(\mathscr{P})_{\mathrm{L}})$.

Proposition 12.21 (Dual of Proposition 12.16). *Assume* $\mathscr{A}$ *has enough projectives and satisfies the condition Ab4. Then the following hold.*

(1) *The canonical functor* $K(\mathscr{A}) \to D(\mathscr{A})$ *induces equivalences* $K(\mathscr{P})_{\mathrm{L}} \xrightarrow{\sim} D(\mathscr{A})$ *and* $K^{+}(\mathscr{P})_{\mathrm{L}} \xrightarrow{\sim} D(\mathscr{A})_{\mathrm{fpd}}$.

(2) *The canonical functors* $K(\mathscr{A}) \to D(\mathscr{A})$ *and* $K(\mathscr{A})_{\mathrm{fpd}} \to D(\mathscr{A})_{\mathrm{fpd}}$ *are colocalizations*.

(3) *The canonical functor* $D(\mathscr{A})_{\mathrm{fpd}} \to D(\mathscr{A})$ *preserves direct sums*.

§13. Right derived functors

Throughout this section, $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are abelian categories and $\mathscr{U}$ is the épaisse subcategory of $K(\mathscr{A})$ consisting of acyclic complexes. Unless otherwise stated, functors are covariant functors.

Definition 13.1. A full triangulated subcategory $K^*(\mathscr{A})$ of $K(\mathscr{A})$ is called a localizing subcategory if the canonical functor

$$K^*(\mathscr{A})/\mathscr{U} \cap K^*(\mathscr{A}) \to D(\mathscr{A})$$

is fully faithful. If $K^*(\mathscr{A})$ is a localizing subcategory of $K(\mathscr{A})$, we denote by $D^*(\mathscr{A})$ the quotient category $K^*(\mathscr{A})/\mathscr{U} \cap K^*(\mathscr{A})$ and by $Q : K^*(\mathscr{A}) \to D^*(\mathscr{A})$ the canonical functor.

Remark 13.1. (1) $K^-(\mathscr{A})$, $K^+(\mathscr{A})$, $K^{-,\,\mathrm{b}}(\mathscr{A})$, $K^{+,\,\mathrm{b}}(\mathscr{A})$ and $K^{\mathrm{b}}(\mathscr{A})$ are localizing subcategories of $K(\mathscr{A})$.

(2) If $\mathscr{A}'$ is a thick subcategory of $\mathscr{A}$, then $K_{\mathscr{A}'}(\mathscr{A})$, $K^-_{\mathscr{A}'}(\mathscr{A})$, $K^+_{\mathscr{A}'}(\mathscr{A})$ and $K^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A})$ are localizing subcategories of $K(\mathscr{A})$.

(3) If $\mathscr{A}$ has enough injectives, then $K^+(\mathscr{A})_{\mathrm{fid}}$ is a localizing subcategory of $K(\mathscr{A})$.

(4) If $\mathscr{A}$ has enough projectives, then $K^-(\mathscr{A})_{\mathrm{fpd}}$ is a localizing subcategory of $K(\mathscr{A})$.

(5) If $K^*(\mathscr{A})$ is a localizing subcategory of $K(\mathscr{A})$, then $K^*(\mathscr{A})^{\mathrm{op}}$ is a localizing subcategory of $K(\mathscr{A})^{\mathrm{op}}$.

(6) We have $K^-(\mathscr{A})^{\mathrm{op}} = K^+(\mathscr{A}^{\mathrm{op}})$, $K^+(\mathscr{A})^{\mathrm{op}} = K^-(\mathscr{A}^{\mathrm{op}})$, $K^{\mathrm{b}}(\mathscr{A})^{\mathrm{op}} = K^{\mathrm{b}}(\mathscr{A}^{\mathrm{op}})$, etc..

Definition 13.2. Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. A right derived functor of F is an initial object of the following category: an object is a pair (ζ, G) of a ∂-functor $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ and $\zeta \in \mathrm{Hom}\,(QF, GQ)$; and a morphism $\eta : (\zeta_1, G_1) \to (\zeta_2, G_2)$ is a morphism $\eta \in \mathrm{Hom}\,(G_1, G_2)$ with $\zeta_2 = \eta_Q \circ \zeta_1$. The right derived functor of F is denoted by $(\cdot, \boldsymbol{R}^*F)$ or simply by $\boldsymbol{R}^*F$. In case $K^*(\mathscr{A}) = K^-(\mathscr{A})$, $K^+(\mathscr{A})$, $K^{\mathrm{b}}(\mathscr{A})$, $K_{\mathscr{A}'}(\mathscr{A})$, etc., $\boldsymbol{R}^*F$ is written $\boldsymbol{R}^-F$, $\boldsymbol{R}^+F$, $\boldsymbol{R}^{\mathrm{b}}F$, $\boldsymbol{R}_{\mathscr{A}'}F$, etc., respectively. If no confusion can result, then $\boldsymbol{R}^*F$ is simply written $\boldsymbol{R}F$. Similarly, a right derived functor of a ∂-functor $F : K^*(\mathscr{A}) \to D(\mathscr{B})$ is defined.

In case $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ is a contravariant ∂-functor, we define a right derived functor of F as a right derived functor of a covariant ∂-functor $F : K^*(\mathscr{A})^{\mathrm{op}} \to K(\mathscr{B})$.

Remark 13.2. Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor which has a right derived functor $(\xi, \boldsymbol{R}^*F)$. Then for any $n \in \mathbb{Z}$ the following hold,

(1) $FT^n : K^*(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $(\xi_{T^n}, \boldsymbol{R}^*F \circ T^n)$.

(2) $T^nF : K^*(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $(T^n\xi, T^n \circ \boldsymbol{R}^*F)$.

Proposition 13.1. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Then the following hold.*

(1) *$QF : K^*(\mathscr{A}) \to D(\mathscr{B})$ vanishes on the acyclic complexes if and only if there exists a ∂-functor $F' : D^*(\mathscr{A}) \to D(\mathscr{B})$ such that $QF \xrightarrow{\sim} F'Q$.*

(2) *If there exist a ∂-functor $F' : D^*(\mathscr{A}) \to D(\mathscr{B})$ and an isomorphism $\xi : QF \xrightarrow{\sim} F'Q$, then (ξ, F') is a right derived functor of F.*

Proof. (1) The "if" part is obvious. The "only if" part follows by Proposition 9.10.

(2) Let $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ be a ∂-functor. Since we have an isomorphism

$$\mathrm{Hom}\,(QF, GQ) \xrightarrow{\sim} \mathrm{Hom}\,(F'Q, GQ),\ \zeta \mapsto \zeta \circ \xi^{-1},$$

it follows by Proposition 9.11 that for any $\zeta \in \mathrm{Hom}\,(QF, GQ)$ there exists a unique $\eta \in \mathrm{Hom}\,(F', G)$ such that $\zeta = \eta_Q \circ \xi$.

Remark 13.3. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$.* Then $T : D^*(\mathscr{A}) \to D^*(\mathscr{A})$ is a right derived functor of $T : K^*(\mathscr{A}) \to K^*(\mathscr{A})$.

Proposition 13.2. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. For a pair $(\xi, \boldsymbol{R}^*F)$ of a ∂-functor $\boldsymbol{R}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$ and a homomorphism of ∂-functors $\xi : QF \to \boldsymbol{R}^*F \circ Q$, the following are equivalent.*

(1) *$(\xi, \boldsymbol{R}^*F)$ is a right derived functor of F.*

(2) *For any ∂-functor $G : D^*(\mathscr{A}) \to D(\mathscr{B})$, the correspondence*

$$\mathrm{Hom}\,(\boldsymbol{R}^*F, G) \to \mathrm{Hom}\,(QF, GQ),\ \eta \mapsto \eta_Q \circ \xi$$

is an isomorphism.

Proof. Obvious.

Corollary 13.3. (1) *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F = (F, \theta) : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume F has a right derived functor $(\xi, \boldsymbol{R}^*F)$ and let $\boldsymbol{R}^*F = (\boldsymbol{R}^*F, \eta)$. Then $T\xi \circ Q\theta = \eta_Q \circ \xi_T$ and, for a homomorphism of ∂-functors $\phi : \boldsymbol{R}^*F \circ T \to T \circ \boldsymbol{R}^*F$, the condition $T\xi \circ Q\theta = \phi_Q \circ \xi_T$ implies $\phi = \eta$.*

(2) *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F, G : K^*(\mathscr{A}) \to K(\mathscr{B})$ ∂-functors. Assume both F and G have right derived functors $(\xi, \boldsymbol{R}^*F)$ and $(\zeta, \boldsymbol{R}^*G)$, respectively. Then we have a correspondence*

$$\operatorname{Hom}(F, G) \to \operatorname{Hom}(\boldsymbol{R}^*F, \boldsymbol{R}^*G), \ \phi \mapsto \boldsymbol{R}^*\phi$$

such that $\zeta \circ Q\phi = (\boldsymbol{R}^*\phi)_Q \circ \xi$.

(3) *Let* $K^{**}(\mathscr{A}) \subset K^*(\mathscr{A})$ *be localizing subcategories of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor. Assume both* F *and* $F|_{K^{**}(\mathscr{A})}$ *have right derived functors* $(\xi, \boldsymbol{R}^*F)$ *and* $(\zeta, \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}))$*, respectively. Then there exists a unique homomorphism of* ∂*-functors*

$$\varphi : \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}) \to \boldsymbol{R}^*F|_{D^{**}(\mathscr{A})}$$

such that $\xi|_{K^{**}(\mathscr{A})} = \varphi_Q \circ \zeta$.

(4) *Let* $K^*(\mathscr{A}) \subset K(\mathscr{A})$, $K^\dagger(\mathscr{B}) \subset K(\mathscr{B})$ *be localizing subcategories and let* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$, $G : K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ *be* ∂*-functors. Assume that* F *has a right derived functor* $(\xi, \boldsymbol{R}^*F)$ *and* $\boldsymbol{R}^*F(D^*(\mathscr{A})) \subset D^\dagger(\mathscr{B})$*, that* G *has a right derived functor* $(\zeta, \boldsymbol{R}^\dagger G)$*, and that* $F(K^*(\mathscr{A})) \subset K^\dagger(\mathscr{B})$ *and* GF *has a right derived functor* $(\psi, \boldsymbol{R}^*(GF))$*. Then there exists a unique homomorphism of* ∂*-functors*

$$\varphi : \boldsymbol{R}^*(GF) \to \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F$$

such that $\boldsymbol{R}^\dagger G(\xi) \circ \zeta_F = \varphi_Q \circ \psi$.

(5) *Let* $\mathscr{A}'$ *be a thick subcategory of* $\mathscr{A}$ *and* $J : D^+(\mathscr{A}') \to D^+(\mathscr{A})$ *the canonical functor. Let* $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ *be a* ∂*-functor. Assume both* F *and* $F|_{K^+(\mathscr{A}')}$ *have right derived functors* $(\xi, \boldsymbol{R}^+F)$ *and* $(\zeta, \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}))$*, respectively. Then there exists a unique homomorphism of* ∂*-functors*

$$\varphi : \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}) \to \boldsymbol{R}^+F \circ J$$

such that $\xi|_{K^+(\mathscr{A}')} = \varphi_Q \circ \zeta$.

Proof. (1) Since $\xi \in \operatorname{Hom}(QF, \boldsymbol{R}^*F \circ Q)$, it follows by definition that $T\xi \circ Q\theta = \eta_Q \circ \xi_T$. Next, let $\phi \in \operatorname{Hom}(\boldsymbol{R}^*F \circ T, T \circ \boldsymbol{R}^*F)$ with $T\xi \circ Q\theta = \phi_Q \circ \xi_T$. Then, since $(\xi_T, \boldsymbol{R}^*F \circ T)$ is a right derived functor of $F \circ T$, $\phi_Q \circ \xi_T = \eta_Q \circ \xi_T$ implies $\phi = \eta$.

(2) $\zeta \circ Q\phi \in \operatorname{Hom}(QF, GQ)$ for all $\phi \in \operatorname{Hom}(F, G)$.

(3) $\xi|_{K^{**}(\mathscr{A})} \in \operatorname{Hom}(Q \circ (F|_{K^{**}(\mathscr{A})}), \boldsymbol{R}^*F|_{D^{**}(\mathscr{A})} \circ Q)$.

(4) $\boldsymbol{R}^\dagger G(\xi) \circ \zeta_F \in \operatorname{Hom}(QGF, \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F \circ Q)$.

(5) $\xi|_{K^+(\mathscr{A}')} \in \operatorname{Hom}(Q \circ (F|_{K^+(\mathscr{A}')}), \boldsymbol{R}^+F \circ J \circ Q)$.

Definition 13.3. Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. In case F has a right derived functor $\boldsymbol{R}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$, we set $\boldsymbol{R}^iF = H^i \circ \boldsymbol{R}^*F : D^*(\mathscr{A}) \to \mathscr{B}$ for $i \in \mathbb{Z}$.

Proposition 13.4. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume F has a right derived functor $\boldsymbol{R}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$. Then for any exact sequence $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ in $C(\mathscr{A})$ with $X^\bullet, Y^\bullet, Z^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ we have a long exact sequence*

$$\cdots \to \boldsymbol{R}^iF(X^\bullet) \to \boldsymbol{R}^iF(Y^\bullet) \to \boldsymbol{R}^iF(Z^\bullet) \to \boldsymbol{R}^{i+1}F(X^\bullet) \to \cdots.$$

Proof. By Proposition 11.1(2).

Lemma 13.5. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$. Assume $K^*(\mathscr{A})$ has a full triangulated subcategory $\mathscr{L}$ such that for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Then for any $u \in K^*(\mathscr{A})(X^\bullet, Y^\bullet)$ and a quasi-isomorphism $s : X^\bullet \to I_X^\bullet$ with $I_X^\bullet \in \mathrm{Ob}(\mathscr{L})$ we have a commutative square*

$$\begin{array}{ccc} X^\bullet & \xrightarrow{u} & Y^\bullet \\ s\downarrow & & \downarrow t \\ I_X^\bullet & \xrightarrow{\hat{u}} & I_Y^\bullet \end{array}$$

in $K^*(\mathscr{A})$ such that t is a quasi-isomorphism with $I_Y^\bullet \in \mathrm{Ob}(\mathscr{L})$. Furthermore, if u is a quasi-isomorphism, so is $\hat{u}$.

Proof. Let $s: X^\bullet \to I_X^\bullet$ be a quasi-isomorphism with $I_X^\bullet \in \mathrm{Ob}(\mathscr{L})$. Form a ∂-square

$$\begin{array}{ccc} X^\bullet & \xrightarrow{u} & Y^\bullet \\ s\downarrow & & \downarrow t' \\ I_X^\bullet & \xrightarrow{u'} & Y'^\bullet \end{array}$$

and take a quasi-isomorphism $t'' : Y'^\bullet \to I_Y^\bullet$ with $I_Y^\bullet \in \mathrm{Ob}(\mathscr{L})$. Since by Lemma 7.4 t' is a quasi-isomorphism, so is $t = t''t' : Y^\bullet \to I_Y^\bullet$. In case u is a quasi-isomorphism, since $\hat{u}s = tu$ is a quasi-isomorphism, so is $\hat{u}$ by Proposition 4.2.

Proposition 13.6 (Existence theorem). *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume $K^*(\mathscr{A})$ has a full triangulated subcategory $\mathscr{L}$ such that*

(1) *for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$, and*

(2) *$QF|_{\mathscr{L}} : \mathscr{L} \to D(\mathscr{B})$ vanishes on the acyclic complexes*.

Then F has a right derived functor $(\xi, \boldsymbol{R}^*F)$ such that $\xi_I : Q(F(I^\bullet)) \to \boldsymbol{R}^*F(Q(I^\bullet))$ is an isomorphism for all $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. In particular, for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, if we take a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$, then $\boldsymbol{R}^iF(Q(X^\bullet)) \cong H^i(F(I^\bullet))$ for all $i \in \mathbb{Z}$.

Proof. Let $J : \mathscr{L} \to K^*(\mathscr{A})$ be the inclusion. By hypothesis (1) and Proposition 8.17(1) the canonical functor $J' : \mathscr{L}/\mathscr{U} \cap \mathscr{L} \to D^*(\mathscr{A})$ is an equivalence. Let $P : D^*(\mathscr{A}) \to \mathscr{L}/\mathscr{U} \cap \mathscr{L}$ be a quasi-inverse of J' and $\varepsilon : \mathbf{1}_{\mathscr{L}/\mathscr{U} \cap \mathscr{L}} \tilde{\to} PJ'$ an isomorphism. Also, by hypothesis (2) and Proposition 9.10 we have a ∂-functor $F' : \mathscr{L}/\mathscr{U} \cap \mathscr{L} \to D(\mathscr{B})$ such that $QFJ = F'Q$. Put $\boldsymbol{R}^*F = F'P$. Let $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ be a ∂-functor. By Proposition 9.11 we have an isomorphism

$$\mathrm{Hom}\,(\boldsymbol{R}^*F, G) \tilde{\to} \mathrm{Hom}\,(QFJ, GQJ),\ \eta \mapsto (\eta_{J'} \circ F'\varepsilon)_Q = \eta_{QJ} \circ F'\varepsilon_Q.$$

We need the following.

Claim: For any ∂-functor $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ we have an isomorphism

$$\mathrm{Hom}\,(QF, GQ) \tilde{\to} \mathrm{Hom}\,(QFJ, GQJ),\ \eta \mapsto \eta_J.$$

Proof. Let $\eta \in \mathrm{Hom}\,(QF, GQ)$ with $\eta_J = 0$. Then, for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, by taking a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$, we get $\eta_X = G(Q(s))^{-1} \circ \eta_I \circ Q(F(s)) = 0$. Thus $\eta = 0$. Conversely, let $\theta \in \mathrm{Hom}\,(QFJ, GQJ)$. For $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, take a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$ and set $\eta_X = G(Q(s))^{-1} \circ \theta_I \circ Q(F(s))$, which does not depend on the choice of s. To see this, take another quasi-isomorphism $s' : X^\bullet \to I'^\bullet$ with $I'^\bullet \in \mathrm{Ob}(\mathscr{L})$. Then by Lemma 13.5 we have a commutative square

$$\begin{array}{ccc} X^\bullet & \xrightarrow{s'} & I'^\bullet \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ I^\bullet & \xrightarrow{t'} & I''^\bullet \end{array}$$

in $K^*(\mathscr{A})$ with t, t' quasi-isomorphisms and $I''^\bullet \in \mathrm{Ob}(\mathscr{L})$. Then we have

$$\begin{aligned} G(Q(s))^{-1} \circ \theta_I \circ Q(F(s)) &= G(Q(s))^{-1} \circ G(Q(t'))^{-1} \circ \theta_{I''} \circ Q(F(t')) \circ Q(F(s)) \\ &= G(Q(t's))^{-1} \circ \theta_{I''} \circ Q(F(t's)) \\ &= G(Q(s'))^{-1} \circ G(Q(t))^{-1} \circ \theta_{I''} \circ Q(F(t)) \circ Q(F(s')) \\ &= G(Q(s'))^{-1} \circ \theta_{I'} \circ Q(F(s')). \end{aligned}$$

Note that $\theta_I = \eta_{JI}$ for all $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Thus it only remains to check that $\eta \in \mathrm{Hom}\,(QF, GQ)$.

Let $u \in K^*(\mathscr{A})(X^\bullet, Y^\bullet)$. By Lemma 13.5 we have a commutative square

$$\begin{array}{ccc} X^\bullet & \xrightarrow{u} & Y^\bullet \\ s\downarrow & & \downarrow t \\ I_X^\bullet & \xrightarrow{\hat{u}} & I_Y^\bullet \end{array}$$

in $K^*(\mathscr{A})$ with s, t quasi-isomorphisms and $I_X^\bullet, I_Y^\bullet \in \mathrm{Ob}(\mathscr{L})$. Since we have a commutative diagram

$$\begin{array}{ccccccc} QF\,X^\bullet & \xrightarrow{QFs} & QF\,I_X^\bullet & \xrightarrow{\theta_{I_X}} & GQ\,I_X^\bullet & \xrightarrow{GQs^{-1}} & GQ\,X^\bullet \\ QFu\downarrow & & \downarrow QF\hat{u} & & \downarrow GQ\hat{u} & & \downarrow GQu \\ QF\,Y^\bullet & \xrightarrow{QFt} & QF\,I_Y^\bullet & \xrightarrow{\theta_{I_Y}} & GQ\,I_Y^\bullet & \xrightarrow{GQt^{-1}} & GQ\,Y^\bullet, \end{array}$$

it follows that $\eta \in \mathrm{Hom}(QF, GQ)$. Next, let $F = (F, \alpha)$ and $G = (G, \beta)$. Then, since $QT = TQ$, by Proposition 7.10(4) $QF = (QF, Q\alpha)$ and $GQ = (GQ, \beta_Q)$. Also, since $JT = TJ$, we have $QFJ = (QFJ, Q\alpha_J)$ and $GQJ = (GQJ, \beta_{QJ})$. Let $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ and take a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Since we have a commutative diagram

$$\begin{array}{ccccccc} QFT\,X^\bullet & \xrightarrow{QFTs} & QFJT\,I^\bullet & \xrightarrow{\theta_{TI}} & GQJT\,I^\bullet & \xrightarrow{GQTs^{-1}} & GQT\,X^\bullet \\ Q\alpha_X\downarrow & & \downarrow Q\alpha_{JI} & & \downarrow Q\beta_{JI} & & \downarrow \beta_{QX} \\ TQF\,X^\bullet & \xrightarrow{TQFs} & TQFJ\,I^\bullet & \xrightarrow{T\theta_I} & TGQJ\,I^\bullet & \xrightarrow{TGQs^{-1}} & TGQ\,X^\bullet, \end{array}$$

it follows that $\eta \in \mathrm{Hom}\,(QF, GQ)$.

By Claim there exists $\xi \in \mathrm{Hom}\,(QF, \boldsymbol{R}^*F \circ Q)$ with $F'\varepsilon_Q = \xi_J$. Then

$$\begin{aligned} (\eta_{J'} \circ F'\varepsilon)_Q &= \eta_{J'Q} \circ F'\varepsilon_Q \\ &= \eta_{QJ} \circ \xi_J \\ &= (\eta_{\overline{Q}} \circ \xi)_J \end{aligned}$$

for all $\eta \in \mathrm{Hom}\,(\boldsymbol{R}^*F, G)$, so that again by Claim we get an isomorphism

$$\mathrm{Hom}\,(\boldsymbol{R}^*F, G) \xrightarrow{\sim} \mathrm{Hom}\,(QF, GQ),\ \eta \mapsto \eta_{\overline{Q}} \circ \xi.$$

Since ε is an isomorphism, so is $\xi_I = F'\varepsilon_{QI}$ for all $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Finally, let $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ and take a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Then

$$\begin{aligned} \boldsymbol{R}^i F(Q(X^\bullet)) &\cong H^i(\boldsymbol{R}^* F(Q(X^\bullet))) \\ &\cong H^i(\boldsymbol{R}^* F(Q(I^\bullet))) \\ &\cong H^i(Q(F(I^\bullet))) \\ &\cong H^i(F(I^\bullet)) \end{aligned}$$

for all $i \in \mathbb{Z}$.

Corollary 13.7. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Assume there exists a subcollection $\mathscr{I}$ of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$,*

(2) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, then $Y \in \mathscr{I}$ if and only if $Z \in \mathscr{I}$, and*

(3) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X, Y, Z \in \mathscr{I}$, then the induced sequence $0 \to FX \to FY \to FZ \to 0$ in $\mathscr{B}$ is exact.*

Then the extended ∂-functor $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $(\xi, \boldsymbol{R}^+ F)$ such that $\xi_I : Q(F(I^\bullet)) \to \boldsymbol{R}^+ F(Q(I^\bullet))$ is an isomorphism for all $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. In particular, for any $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ in $K^+(\mathscr{A})$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ and $\boldsymbol{R}^i F(X^\bullet) \cong H^i(F(I^\bullet))$ for all $i \in \mathbb{Z}$.

Proof. Note that by Proposition 6.1(2) $K^+(\mathscr{I})$ is a full triangulated subcategory of $K^+(\mathscr{A})$. The following enables us to apply Proposition 13.6 for $\mathscr{L} = K^+(\mathscr{I})$.

Claim: (1) For any $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

(2) $QF : K^+(\mathscr{I}) \to D(\mathscr{B})$ vanishes on the acyclic complexes.

Proof. (1) By hypothesis (1) and Proposition 4.7.

(2) Let $I^\bullet \in \mathrm{Ob}(\mathscr{U} \cap K^+(\mathscr{I}))$. By hypothesis (2) $Z^i(I^\bullet) \in \mathscr{I}$ for all $i \in \mathbb{Z}$. Thus by hypothesis (3) $F(I^\bullet)$ is acyclic.

Remark 13.4. In Corollary 13.7, the following hold.

(1) $\boldsymbol{R}^i F(X) = 0$ for all $X \in \mathrm{Ob}(\mathscr{A})$ and $i < 0$.

(2) $\boldsymbol{R}^i F(I) = 0$ for all $I \in \mathscr{I}$ and i 0.

(3) $\boldsymbol{R}^0 F : \mathscr{A} \to \mathscr{B}$ is left exact.

(4) We have a homomorphism $\varphi : F \to \boldsymbol{R}^0 F$ such that φ_I is an isomorphism for all $I \in \mathscr{I}$ and $F : \mathscr{A} \to \mathscr{B}$ is left exact if and only if φ is an isomorphism.

Proposition 13.8. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$*

a ∂-functor. Assume the canonical functor $Q : K^*(\mathscr{A}) \to D^*(\mathscr{A})$ *has a right adjoint* $P : D^*(\mathscr{A}) \to K^*(\mathscr{A})$ *and let* $\varepsilon : \mathbf{1}_{K^*(\mathscr{A})} \to PQ$ *be the unit. Then the following hold.*

(1) *F has a right derived functor* $(\xi, \boldsymbol{R}^*F)$ *with* $\boldsymbol{R}^*F = QFP$ *and* $\xi = QF\varepsilon$.

(2) *Assume P is fully faithful. Then for any* $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$, ξ_{PX} *is an isomorphism and* $\boldsymbol{R}^iF(X^\bullet) \cong H^i(FP X^\bullet)$ *for all* $i \in \mathbb{Z}$.

Proof. (1) Let $\delta : QP \to \mathbf{1}_{D^*(\mathscr{A})}$ be the counit. Let $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ be a ∂-functor and define correspondences

$$\alpha_G : \mathrm{Hom}(QFP, G) \to \mathrm{Hom}(QF, GQ),\ \eta \mapsto \eta_Q \circ QF\varepsilon,$$

$$\beta_G : \mathrm{Hom}(QF, GQ) \to \mathrm{Hom}(QFP, G),\ \zeta \mapsto G\delta \circ \zeta_P.$$

Claim: (a) α_G is an isomorphism with $\alpha_G^{-1} = \beta_G$.

(b) $\alpha_G(\eta) \in \mathrm{Hom}\,(QF, GQ)$ for all $\eta \in \mathrm{Hom}\,(QFP, G)$.

(c) $\beta_G(\zeta) \in \mathrm{Hom}\,(QFP, G)$ for all $\zeta \in \mathrm{Hom}\,(QF, GQ)$.

Proof. (a) By the fact that $P\delta \circ \varepsilon_P = \mathrm{id}_P$ and $\delta_Q \circ Q\varepsilon = \mathrm{id}_Q$.

(b) By Proposition 12.1 $\varepsilon \in \mathrm{Hom}\,(\mathbf{1}_{K^*(\mathscr{A})}, PQ)$. Thus $QF\varepsilon \in \mathrm{Hom}\,(QF, QFPQ)$, so that $\eta_Q \circ QF\varepsilon \in \mathrm{Hom}\,(QF, GQ)$ for all $\eta \in \mathrm{Hom}\,(QFP, G)$.

(c) By Proposition 12.1 $\delta \in \mathrm{Hom}\,(QP, \mathbf{1}_{D^*(\mathscr{A})})$. Thus $G\delta \in \mathrm{Hom}\,(GQP, G)$, so that $G\delta \circ \zeta_P \in \mathrm{Hom}\,(QFP, G)$ for all $\zeta \in \mathrm{Hom}\,(QF, GQ$.

Consequently, we get an isomorphism

$$\mathrm{Hom}\,(QFP, G) \to \mathrm{Hom}\,(QF, GQ),\ \eta \mapsto \eta_Q \circ QF\varepsilon$$

and Proposition 13.2 applies.

(2) Assume P is fully faithful. Then δ is an isomorphism. Thus, since $P\delta \circ \varepsilon_P = \mathrm{id}_P$, ε_P is an isomorphism, so is ξ_P. Finally, for any $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$ and $i \in \mathbb{Z}$, we have $\boldsymbol{R}^iF(X^\bullet) = H^i(\boldsymbol{R}^*F(X^\bullet)) = H^i(Q(F(P(X^\bullet))) = H^i(F(P(X^\bullet)))$.

Corollary 13.9. *Let* $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ *be a ∂-functor. Assume* $\mathscr{A}$ *has enough injectives and let* $\mathscr{I}$ *be the collection of injective objects of* $\mathscr{A}$*. Then the following hold.*

(1) *F has a right derived functor* $(\xi, \boldsymbol{R}^+F)$ *such that* $\xi_I : Q(F(I^\bullet)) \to \boldsymbol{R}^+F(Q(I^\bullet))$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

(2) *For any* $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$, *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *in* $K^+(\mathscr{A})$ *with* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ *and* $\boldsymbol{R}^iF(X^\bullet) \cong H^i(F(I^\bullet))$ *for all* $i \in \mathbb{Z}$.

(3) *If F is given by an additive functor* $F : \mathscr{A} \to \mathscr{B}$, *then the functor* $\boldsymbol{R}^iF|_{\mathscr{A}} : \mathscr{A} \to \mathscr{B}$

coincides with the usual i^{th} right derived functor of $F : \mathscr{A} \to \mathscr{B}$ for all $i \geq 0$.

Proof. (1) and (2) By Propositions 12.4 and 13.8.

(3) Let $X \in \mathrm{Ob}(\mathscr{A})$ and $X \to I^\bullet$ an injective resolution of X. Then $X \cong I^\bullet$ in $D^+(\mathscr{A})$ and thus $\boldsymbol{R}^iF(X) \cong \boldsymbol{R}^iF(I^\bullet)$ for all $i \in \mathbb{Z}$.

Corollary 13.10. *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$ and $F : K^+_{\mathscr{A}'}(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume $\mathscr{A}'$ has enough $\mathscr{A}$-injectives and let $\mathscr{I}$ be the collection of injective objects of $\mathscr{A}$. Then the following hold.*

(1) *F has a right derived functor $(\xi, \boldsymbol{R}^+_{\mathscr{A}'}F)$ such that $\xi_I : Q(F(I^\bullet)) \to \boldsymbol{R}^+_{\mathscr{A}'}F(Q(I^\bullet))$ is an isomorphism for all $I^\bullet \in K^+(\mathscr{I} \cap \mathscr{A}')$.*

(2) *For any $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ in $K^+_{\mathscr{A}'}(\mathscr{A})$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I} \cap \mathscr{A}'))$ and $\boldsymbol{R}^iF(X^\bullet) \cong H^i(F(I^\bullet))$ for all $i \in \mathbb{Z}$.*

Proof. By Propositions 12.5 and 13.8.

Proposition 13.11. *Let $K^{**}(\mathscr{A}) \subset K^*(\mathscr{A})$ be localizing subcategories of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume $K^*(\mathscr{A})$ has a full triangulated subcategory $\mathscr{L}$ such that*

(1) *for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$,*

(2) *for any $X^\bullet \in \mathrm{Ob}(K^{**}(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L} \cap K^{**}(\mathscr{A}))$, and*

(3) *$QF|_{\mathscr{L}} : \mathscr{L} \to D(\mathscr{B})$ vanishes on the acyclic complexes.*

Then both F and $F|_{K^{**}(\mathscr{A})}$ have right derived functors $(\xi, \boldsymbol{R}^*F)$ and $(\zeta, \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}))$, respectively, and the canonical homomorphism

$$\varphi : \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}) \to \boldsymbol{R}^*F|_{D^{**}(\mathscr{A})}$$

is an isomorphism.

Proof. By Proposition 13.6 both F and $F|_{K^{**}(\mathscr{A})}$ have right derived functors $(\xi, \boldsymbol{R}^*F)$ and $(\zeta, \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}))$, respectively, and by Corollary 13.3(3) we have a unique homomorphism of ∂-functors

$$\varphi : \boldsymbol{R}^{**}(F|_{K^{**}(\mathscr{A})}) \xrightarrow{\sim} \boldsymbol{R}^*F|_{D^{**}(\mathscr{A})}$$

such that $\xi|_{K^{**}(\mathscr{A})} = \varphi_Q \circ \zeta$. For any $I^\bullet \in \mathrm{Ob}(\mathscr{L} \cap K^{**}(\mathscr{A}))$, by Proposition 13.6 both ξ_I and ζ_I are isomorphisms, so that φ_{QI} is an isomorphism. Thus, since by hypothesis (2) the canonical functor $Q : \mathscr{L} \cap K^{**}(\mathscr{A}) \to D^{**}(\mathscr{A})$ is dense, φ is an isomorphism.

Proposition 13.12. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Let $K^\dagger(\mathscr{B})$ be a localizing subcategory of $K(\mathscr{B})$ and $G : K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ a ∂-functor. Assume*

(1) *$K^*(\mathscr{A})$ has a full triangulated subcategory $\mathscr{L}$ for which the hypotheses (1), (2) of Proposition 13.6 are satisfied,*

(2) *$K^\dagger(\mathscr{B})$ has a full triangulated subcategory $\mathscr{M}$ for which the hypotheses (1), (2) of Proposition 13.6 are satisfied, and*

(3) *$F(K^*(\mathscr{A})) \subset K^\dagger(\mathscr{B})$ and $F(\mathscr{L}) \subset \mathscr{M}$.*

Then F, G and GF have right derived functors $(\xi, \boldsymbol{R}^*F)$, $(\zeta, \boldsymbol{R}^\dagger G)$ and $(\psi, \boldsymbol{R}^*(GF))$, respectively, $\boldsymbol{R}^*F(D^*(\mathscr{A})) \subset D^\dagger(\mathscr{B})$, and the canonical homomorphism

$$\varphi : \boldsymbol{R}^*(GF) \to \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F$$

is an isomorphism.

Proof. By Proposition 13.6 F and G have right derived functors $(\xi, \boldsymbol{R}^*F)$ and $(\zeta, \boldsymbol{R}^\dagger G)$, respectively. Let $X^\bullet \in \mathrm{Ob}(\mathscr{L})$ be acyclic. Then, since $Q(F(X^\bullet)) = 0$, by Proposition 9.3(3) $F(X^\bullet)$ is acyclic and $Q(G(F(X^\bullet))) = 0$. Thus, again by Proposition 13.6 GF has a right derived functor $(\psi, \boldsymbol{R}^*(GF))$. Also, for any $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$, since we have a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{L})$, $\boldsymbol{R}^*F(X^\bullet) \cong \boldsymbol{R}^*F(Q(I^\bullet)) \cong Q(F(I^\bullet)) \in \mathrm{Ob}(D^\dagger(\mathscr{B}))$. Thus by Corollary 13.3(4) we have a unique homomorphism of ∂-functors

$$\varphi : \boldsymbol{R}^*(GF) \tilde{\to} \boldsymbol{R}^\dagger G \circ \boldsymbol{R}^*F$$

such that $\boldsymbol{R}^\dagger G(\xi) \circ \zeta_F = \varphi_Q \circ \psi$. Let $I^\bullet \in \mathrm{Ob}(\mathscr{L})$. Then by Proposition 13.6 ξ_I, ζ_{FI} and ψ_I are isomorphisms, so that φ_{QI} is an isomorphism. Thus, since $Q : \mathscr{L} \to D^*(\mathscr{A})$ is dense, φ is an isomorphism.

Proposition 13.13. *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$ and $J : D^+(\mathscr{A}') \to D^+(\mathscr{A})$ the canonical functor. Let $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ be a ∂-functor. Assume $\mathscr{A}$ has enough injectives and $\mathscr{A}'$ has enough $\mathscr{A}$-injectives. Then both F and $F|_{K^+(\mathscr{A}')}$ have right derived functors $(\xi, \boldsymbol{R}^+F)$ and $(\zeta, \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}))$, respectively, and the canonical homomorphism*

$$\varphi : \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}) \to \boldsymbol{R}^+F \circ J$$

is an isomorphism.

Proof. By Corollary 13.9(1) both F and $F|_{K^+(\mathscr{A}')}$ have right derived functors $(\xi, \boldsymbol{R}^+F)$ and

$(\zeta, \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}))$, respectively. Thus by Corollary 13.3(5) we have a unique homomorphism of ∂-functors

$$\varphi : \boldsymbol{R}^+(F|_{K^+(\mathscr{A}')}) \xrightarrow{\sim} \boldsymbol{R}^+F \circ J$$

such that $\xi|_{K^+(\mathscr{A}')} = \varphi_Q \circ \zeta$. Let $\mathscr{I}$ be the collection of injective objects of $\mathscr{A}$. Let $I^\bullet \in \operatorname{Ob}(\mathscr{I} \cap \mathscr{A}')$. Then by Corollary 13.9(1) both ξ_I and ζ_I are isomorphisms, so that φ_{QI} is an isomorphism. Thus, since by Corollary 13.9(2) $Q : K^+(\mathscr{I} \cap \mathscr{A}') \to D^+(\mathscr{A}')$ is dense, φ is an isomorphism.

Proposition 13.14. *Assume $\mathscr{A}$ has enough injectives. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor which has an exact left adjoint $U : \mathscr{B} \to \mathscr{A}$. Then the following hold.*

(1) *The extended ∂-functor $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $(\xi, \boldsymbol{R}^+F)$ which satisfies $\boldsymbol{R}^+F(D^+(\mathscr{A})) \subset D^+(\mathscr{B})$.*

(2) *For the extended ∂-functor $U : K^+(\mathscr{B}) \to K(\mathscr{A})$, there exists a ∂-functor $\overline{U} : D^+(\mathscr{B}) \to D(\mathscr{A})$ such that $QU = \overline{U}Q$.*

(3) *$\overline{U} : D^+(\mathscr{B}) \to D^+(\mathscr{A})$ is a left adjoint of $\boldsymbol{R}^+F : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*

(4) *If F is fully faithful, so is $\boldsymbol{R}^+F$.*

Proof. (1) By Corollary 13.7.

(2) It is obvious that $QU : K^+(\mathscr{B}) \to D(\mathscr{A})$ vanishes on the acyclic complexes.

(3) By Proposition 3.10 $U : K^+(\mathscr{B}) \to K^+(\mathscr{A})$ is a left adjoint of $F : K^+(\mathscr{A}) \to K^+(\mathscr{B})$. Let $\varepsilon : \mathbf{1}_{K^+(\mathscr{B})} \to FU$, $\delta : UF \to \mathbf{1}_{K^+(\mathscr{A})}$ be the unit and the counit, respectively.

Claim 1: There exists $\theta \in \operatorname{Hom}(\overline{U} \circ \boldsymbol{R}^+F, \mathbf{1}_{D^+(\mathscr{A})})$ such that $Q\delta = \theta_Q \circ \overline{U}\xi$.

Proof. Let $\mathscr{I}$ be the collection of injective objects of $\mathscr{A}$. For any $X^\bullet \in \operatorname{Ob}(K^+(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \operatorname{Ob}(K^+(\mathscr{I}))$. Also, ξ_I is an isomorphism for all $I^\bullet \in \operatorname{Ob}(K^+(\mathscr{I}))$. For $X^\bullet \in \operatorname{Ob}(K^+(\mathscr{A}))$, take a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \operatorname{Ob}(K^+(\mathscr{I}))$ and set $\overline{\theta}_X = Q(s)^{-1} \circ Q\delta_I \circ \overline{U}\xi_I^{-1} \circ \overline{U}(\boldsymbol{R}^+F(Q(s)))$, which does not depend on the choice of s. Then $\overline{\theta} \in \operatorname{Hom}(\overline{U} \circ \boldsymbol{R}^+F \circ Q, Q)$ and $Q\delta = \overline{\theta} \circ \overline{U}\xi$. Thus by Proposition 9.11 there exists $\theta \in \operatorname{Hom}(\overline{U} \circ \boldsymbol{R}^+F, \mathbf{1}_{D^+(\mathscr{A})})$ such that $\overline{\theta} = \theta_Q$.

Claim 2: There exists $\eta \in \operatorname{Hom}(\mathbf{1}_{D^+(\mathscr{B})}, \boldsymbol{R}^+F \circ \overline{U})$ such that $\xi_U \circ Q\varepsilon = \eta_Q$.

Proof. By Proposition 9.11.

Claim 3: $\boldsymbol{R}^+F\theta \circ \eta_{\boldsymbol{R}^+F} = \mathrm{id}_{\boldsymbol{R}^+F}$.

Proof. We have

$$
\begin{aligned}
(\boldsymbol{R}^{+}F\theta \circ \eta_{\boldsymbol{R}^{+}F})_{Q} \circ \xi &= \boldsymbol{R}^{+}F\theta_{Q} \circ \eta_{\boldsymbol{R}^{+}FQ} \circ \xi \\
&= \boldsymbol{R}^{+}F\theta_{Q} \circ \boldsymbol{R}^{+}F\overline{U}\xi \circ \eta_{QF} \\
&= \boldsymbol{R}^{+}F(\theta_{Q} \circ \overline{U}\xi) \circ \eta_{QF} \\
&= \boldsymbol{R}^{+}F(Q\delta) \circ \xi_{UF} \circ Q\varepsilon_{F} \\
&= \xi \circ QF\delta \circ Q\varepsilon_{F} \\
&= \xi.
\end{aligned}
$$

Thus by Proposition 13.2 $\boldsymbol{R}^{+}F\theta \circ \eta_{\boldsymbol{R}^{+}F} = \mathrm{id}_{\boldsymbol{R}^{+}F}$.

Claim 4: $\theta_{\overline{U}} \circ \overline{U}\eta = \mathrm{id}_{\overline{U}}$.

Proof. We have

$$
\begin{aligned}
(\theta_{\overline{U}} \circ \overline{U}\eta)_{Q} &= \theta_{\overline{U}Q} \circ \overline{U}\eta_{Q} \\
&= \theta_{QU} \circ \overline{U}\xi_{U} \circ \overline{U}Q\varepsilon \\
&= (\theta_{Q} \circ \overline{U}\xi)_{U} \circ QU\varepsilon \\
&= Q\delta_{U} \circ QU\varepsilon \\
&= \mathrm{id}_{QU}.
\end{aligned}
$$

Thus by Proposition 9.11 $\theta_{\overline{U}} \circ \overline{U}\eta = \mathrm{id}_{\overline{U}}$.

(4) Assume δ is an isomorphism. For any $I^{\bullet} \in \mathrm{Ob}(K^{+}(\mathcal{I}))$, since $Q\delta = \theta_{Q} \circ \overline{U}\xi$, and since ξ_{I} is an isomorphism, θ_{QI} is an isomorphism. Since $K^{+}(\mathcal{I}) \xrightarrow{\sim} D^{+}(\mathcal{A})$, it follows that θ is an isomorphism.

§14. Left derived functors

Throughout this section, $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are abelian categories, $\mathscr{U}$ is the épaisse subcategory of $K(\mathscr{A})$ consisting of acyclic complexes. Unless otherwise stated, functors are covariant functors.

Definition 14.1. Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. A left derived functor of F is a terminal object of the following category: an object is a pair (G, ζ) of a ∂-functor $G : D^*(\mathscr{A}) \to D(\mathscr{B})$ and $\zeta \in \mathrm{Hom}\,(GQ, QF)$; and a morphism $\eta : (G_1, \zeta_1) \to (G_2, \zeta_2)$ is a morphism $\eta \in \mathrm{Hom}\,(G_1, G_2)$ with $\zeta_1 = \zeta_2 \circ \eta_Q$. The left derived functor of F is denoted by $(\boldsymbol{L}^*F, \cdot)$ or simply by $\boldsymbol{L}^*F$. In case $K^*(\mathscr{A}) = K^-(\mathscr{A}), K^+(\mathscr{A}), K^b(\mathscr{A})$, $K_{\mathscr{A}'}(\mathscr{A})$, etc., $\boldsymbol{L}^*F$ is written $\boldsymbol{L}^-F$, $\boldsymbol{L}^+F$, $\boldsymbol{L}^bF$, $\boldsymbol{L}_{\mathscr{A}'}F$, etc., respectively. If no confusion can result, then $\boldsymbol{L}^*F$ is simply written $\boldsymbol{L}F$. Similarly, a left derived functor of a ∂-functor $F : K^*(\mathscr{A}) \to D(\mathscr{B})$ is defined.

In case $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ is a contravariant ∂-functor, by replacing $K^*(\mathscr{A})$ with the opposite category $K^*(\mathscr{A})^{\mathrm{op}}$, we define a left derived functor of F as a left derived functor of a covariant ∂-functor $F : K^*(\mathscr{A})^{\mathrm{op}} \to K(\mathscr{B})$.

Remark 14.1. Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor which has a left derived functor $(\boldsymbol{L}^*F, \xi)$. Then for any $n \in \mathbb{Z}$ the folowing hold.

(1) $FT^n : K^*(\mathscr{A}) \to K(\mathscr{B})$ has a left derived functor $(\boldsymbol{L}^*F \circ T^n, \xi_{T^n})$.

(2) $T^nF : K^*(\mathscr{A}) \to K(\mathscr{B})$ has a left derived functor $(T^n \circ \boldsymbol{L}^*F, T^n\xi)$.

Proposition 14.1 (Dual of Proposition 13.1). *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Then the following hold.*

(1) *$QF : K^*(\mathscr{A}) \to D(\mathscr{B})$ vanishes on the acyclic complexes if and only if there exists a ∂-functor $F' : D^*(\mathscr{A}) \to D(\mathscr{B})$ such that $F'Q \xrightarrow{\sim} QF$.*

(2) *If there exist a ∂-functor $F' : D^*(\mathscr{A}) \to D(\mathscr{B})$ and an isomorphism $\xi : F'Q \xrightarrow{\sim} QF$, then (F', ξ) is a left derived functor of F.*

Remark 14.2. *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$.* Then $T : D^*(\mathscr{A}) \to D^*(\mathscr{A})$ is a left derived functor of $T : K^*(\mathscr{A}) \to K^*(\mathscr{A})$.

Proposition 14.2 (Dual of Proposition 13.2). *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. For a pair $(\boldsymbol{L}^*F, \xi)$ of a ∂-functor $\boldsymbol{L}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$ and a homomorphism of ∂-functors $\xi : \boldsymbol{L}^*F \circ Q \to QF$, the following are equivalent.*

(1) *$(\boldsymbol{L}^*F, \xi)$ is a left derived functor of F.*

(2) *For any ∂-functor $G : D^*(\mathscr{A}) \to D(\mathscr{B})$, the correspondence*

$$\mathrm{Hom}\,(G, \boldsymbol{L}^*F) \to \mathrm{Hom}\,(GQ, QF),\ \eta \mapsto \xi \circ \eta_Q$$

is an isomorphism.

Corollary 14.3 (Dual of Corollary 13.3). (1) *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and $F = (F, \theta) : K^*(\mathscr{A}) \to K(\mathscr{B})$ a ∂-functor. Assume F has a left derived functor $(\boldsymbol{L}^*F, \xi)$. Then $Q\theta \circ \xi_T = T\xi \circ \eta_Q$ and, for a homomorphism of ∂-functors $\phi : \boldsymbol{L}^*F \circ T \to T \circ \boldsymbol{L}^*F$, the condition $Q\theta \circ \xi_T = T\xi \circ \phi_Q$ implies $\phi = \eta$.*

(2) *Let $K^*(\mathscr{A})$ be a localizing subcategory of $K(\mathscr{A})$ and F, $G : K^*(\mathscr{A}) \to K(\mathscr{B})$ ∂-functors. Assume both F and G have left derived functors $(\boldsymbol{L}^*F, \xi)$ and $(\boldsymbol{L}^*G, \zeta)$, respectively. Then we have a correspondence*

$$\mathrm{Hom}\,(F, G) \to \mathrm{Hom}\,(\boldsymbol{L}^*F, \boldsymbol{L}^*G),\ \ \phi \mapsto \boldsymbol{L}^*\phi$$

such that $Q\phi \circ \xi = \zeta \circ (\boldsymbol{L}^*\phi)_Q$.

(3) *Let $K^{**}(\mathscr{A}) \subset K^*(\mathscr{A})$ be localizing subcategories of $K(\mathscr{A})$ and $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ a -functor . Assume both F and $F\,|_{K^{**}(\mathscr{A})}$ have left derived functors $(\boldsymbol{L}^*F, \xi)$ and $(\boldsymbol{L}^{**}(F\,|_{K^{**}(\mathscr{A})}), \zeta)$, respectively. Then there exists a unique homomorphism of ∂-functors*

$$\varphi : \boldsymbol{L}^*F\ |_{D^{**}(\mathscr{A})} \to \boldsymbol{L}^{**}(F\,|_{K^{**}(\mathscr{A})})$$

such that $\xi\,|_{K^{**}(\mathscr{A})} = \zeta \circ \varphi_Q$.

(4) *Let $K^*(\mathscr{A}) \subset K(\mathscr{A})$, $K^\dagger(\mathscr{B}) \subset K(\mathscr{B})$ be localizing subcategories and let $F : K^*(\mathscr{A}) \to K(\mathscr{B})$, $G : K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ be ∂-functors. Assume that F has a left derived functor $(\boldsymbol{L}^*F, \xi)$ and $\boldsymbol{L}^*F(D^*(\mathscr{A})) \subset D^\dagger(\mathscr{B})$, that G has a left derived functor $(\boldsymbol{L}^\dagger F, \zeta)$, and that $F(K^*(\mathscr{A})) \subset K^\dagger(\mathscr{B})$ and GF has a left derived functor $(\boldsymbol{L}^*(GF), \psi)$. Then there exists a unique homomorphism of ∂-functors*

$$\varphi : \boldsymbol{L}^\dagger G \circ \boldsymbol{L}^*F \to \boldsymbol{L}^*(GF)$$

such that $\zeta_F \circ \boldsymbol{L}^\dagger G(\xi) = \psi \circ \varphi_Q$.

(5) *Let $\mathscr{A}'$ be a thick subcategory of $\mathscr{A}$ and $J : D^-(\mathscr{A}') \to D^-(\mathscr{A})$ the canonical functor. Let $F : K^-(\mathscr{A}) \to K(\mathscr{B})$ be a ∂-functor. Assume both F and $F\,|_{K^-(\mathscr{A}')}$ have left derived functors $(\boldsymbol{L}^-F, \xi)$ and $(\zeta, \boldsymbol{L}^-(F\,|_{K^-(\mathscr{A}')}))$, respectively. Then there exists a unique homomorphism of ∂-functors*

$$\varphi : \boldsymbol{L}^-F \circ J \to \boldsymbol{L}^-(F\,|_{K^-(\mathscr{A}')})$$

such that $\xi\,|_{K^-(\mathscr{A}')} = \zeta \circ \varphi_Q$.

Definition 14.2. *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor*. If F has a left derived functor $\boldsymbol{L}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$, we set $\boldsymbol{L}_iF = H^{-i} \circ \boldsymbol{L}^*F : D^*(\mathscr{A}) \to \mathscr{B}$ for $i \in \mathbb{Z}$.

Proposition 14.4 (Dual of Proposition 13.4). *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor. Assume* F *has a left derived functor* $\boldsymbol{L}^*F : D^*(\mathscr{A}) \to D(\mathscr{B})$. *Then for an exact sequence* $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ *in* $C(\mathscr{A})$ *with* $X^\bullet, Y^\bullet, Z^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, *we have a long exact sequence*

$$\cdots \to \boldsymbol{L}_iF(X^\bullet) \to \boldsymbol{L}_iF(Y^\bullet) \to \boldsymbol{L}_iF(Z^\bullet) \to \boldsymbol{L}_{i-1}F(X^\bullet) \to \cdots.$$

Lemma 14.5 (Dual of Lemma 13.5). *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$. *Assume* $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$ *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$. *Then for any* $u \in K^*(\mathscr{A})(X^\bullet, Y^\bullet)$ *and a quasi-isomorphism* $s : P_X^\bullet \to X^\bullet$ *with* $P_X^\bullet \in \mathrm{Ob}(\mathscr{L})$ *we have a commutative square*

$$\begin{array}{ccc} P_X^\bullet & \xrightarrow{\hat{u}} & P_Y^\bullet \\ s\downarrow & & \downarrow t \\ X^\bullet & \xrightarrow{u} & Y^\bullet \end{array}$$

in $K^*(\mathscr{A})$ *such that* t *is a quasi-isomorphism with* $P_Y^\bullet \in \mathrm{Ob}(\mathscr{L})$. *Furthermore, if* u *is a quasi-isomorphism, so is* $\hat{u}$.

Proposition 14.6 (Dual of Proposition 13.6). *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor*. *Assume* $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that*

(1) *for any* $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$, *and*

(2) $QF|_{\mathscr{L}} : \mathscr{L} \to D(\mathscr{B})$ *vanishes on the acyclic complexes*.

Then F *has a left derived functor* $(\boldsymbol{L}^*F, \xi)$ *such that* $\xi_P : \boldsymbol{L}^*F(Q(P^\bullet)) \to Q(F(P^\bullet))$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$. *In particular, for any* $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$, *if we take a quasi-isomorphism* $P^\bullet \to X^\bullet$ *in* $K^*(\mathscr{A})$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$, *then* $\boldsymbol{R}^iF(X^\bullet) \cong H^i(F(I^\bullet))$ *for all* $i \in \mathbb{Z}$.

Corollary 14.7 (Dual of Corollary 13.7). *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an additive functor*. *Assume there exists a subcollection* $\mathscr{P}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists an epimorphism* $P \to X$ *in* $\mathscr{A}$ *with* $P \in \mathscr{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $Z \in \mathscr{P}$, *then* $Y \in \mathscr{P}$ *if and only if* $X \in \mathscr{P}$, *and*

(3) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{P}$, *then the induced sequence* $0 \to FX \to FY \to FZ \to 0$ *is exact.*

Then the induced ∂*-functor* $F : K^-(\mathscr{A}) \to K(\mathscr{B})$ *has a left derived functor* $(\boldsymbol{L}^- F, \xi)$ *such that* $\xi_P : \boldsymbol{L}^- F(Q(P^\bullet)) \to Q(F(P^\bullet))$ *is an isomorphism for every* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{I}))$. *In particular, for any* $X^\bullet \in \mathrm{Ob}(D^-(\mathscr{A}))$, *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *in* $K^-(\mathscr{A})$ *with* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{I}))$ *and* $\boldsymbol{L}_i F(X^\bullet) \cong H^{-i}(F(P^\bullet))$ *for all* $i \in \mathbb{Z}$.

Remark 14.3. In Corollary 14.6, the following hold.

(1) $\boldsymbol{L}_i F(X) = 0$ for all $X \in \mathrm{Ob}(\mathscr{A})$ and $i < 0$.

(2) $\boldsymbol{L}_i F(P) = 0$ for all $P \in \mathscr{P}$ for all $i \in \mathbb{Z}$.

(3) $\boldsymbol{L}_0 F : \mathscr{A} \to \mathscr{B}$ is right exact.

(4) We have a homomorphism $\varphi : \boldsymbol{L}_0 F \to F$ such that φ_P is an isomorphism for all $P \in \mathscr{P}$ and $F : \mathscr{A} \to \mathscr{B}$ is right exact if and only if φ is an isomorphism.

Proposition 14.8 (Dual of Proposition 13.8). *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor*. *Assume the canonical functor* $Q : K^*(\mathscr{A}) \to D^*(\mathscr{A})$ *has a left adjoint* $P : D^*(\mathscr{A}) \to K^*(\mathscr{A})$ *and let* $\delta : PQ \to \mathbf{1}_{K^*(\mathscr{A})}$ *be the counit. Then the following hold*.

(1) F *has a left derived functor* $(\boldsymbol{L}^* F, \xi)$ *such that* $\boldsymbol{L}^* F = QFP$ *and* $\xi = QF\delta$.

(2) *Assume* P *is fully faithful. Then for any* $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$, ξ_{PX} *is an isomorphism and* $\boldsymbol{L}_i F(X^\bullet) \cong H^{-i}(F(P(X^\bullet)))$ *for all* $i \in \mathbb{Z}$.

Corollary 14.9 (Dual of Corollary 13.9). *Let* $F : K^-(\mathscr{A}) \to K(\mathscr{B})$ *be a* ∂*-functor. Assume* $\mathscr{A}$ *has enough projectives and let* $\mathscr{P}$ *be the collection of projective objects of* $\mathscr{A}$. *Then the following hold*.

(1) F *has a left derived functor* $(\boldsymbol{L}^- F, \xi)$ *such that* $\xi_P : \boldsymbol{L}^- F(Q(P^\bullet)) \to Q(F(P^\bullet))$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$.

(2) *For any* $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$, *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *in* $K^-(\mathscr{A})$ *with* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$ *and* $\boldsymbol{L}_i F(X^\bullet) \cong H^{-i}(F(P^\bullet))$ *for all* $i \in \mathbb{Z}$.

(3) *If* F *is given by an additive functor* $F : \mathscr{A} \to \mathscr{B}$, *then the functor* $\boldsymbol{L}_i F\,|_{\mathscr{A}} : \mathscr{A} \to \mathscr{B}$ *coincides with the usual* i^{th} *left derived functor of* $F : \mathscr{A} \to \mathscr{B}$ *for all* $i \geq 0$.

Corollary 14.10 (Dual of Corollary 13.10). *Let* $\mathscr{A}$*' be a thick subcategory of* $\mathscr{A}$ *and let* $F : K^-_{\mathscr{A}'}(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor. Assume* $\mathscr{A}$*' has enough* $\mathscr{A}$*-projectives and let* $\mathscr{P}$ *be the collection of projective objects of* $\mathscr{A}$. *Then*

(1) F *has a left derived functor* $(\boldsymbol{L}^-_{\mathscr{A}'} F, \xi)$ *such that* $\xi_P : \boldsymbol{L}^-_{\mathscr{A}'} F(Q(P^\bullet)) \to Q(F(P^\bullet))$ *is an isomorphism for all* $P^\bullet \in K^+(\mathscr{P} \cap \mathscr{A}')$.

(2) *For any* $X^\bullet \in \mathrm{Ob}(D^-_{\mathscr{A}'}(\mathscr{A}))$, *we have a quasi-isomorphism* $P^\bullet \to X^\bullet$ *in* $K^-_{\mathscr{A}'}(\mathscr{A})$ *with* $P^\bullet \in K^+(\mathscr{P} \cap \mathscr{A}')$ *and* $\boldsymbol{L}_i F(X^\bullet) \cong H^{-i}(F(P^\bullet))$ *for all* $i \in \mathbb{Z}$.

Proposition 14.11 (Dual of Proposition 13.11). *Let* $K^{**}(\mathscr{A}) \subset K^*(\mathscr{A})$ *be localizing subcategories of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor. Assume* $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that*

(1) *for any* $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$,

(2) *for any* $X^\bullet \in \mathrm{Ob}(K^{**}(\mathscr{A}))$, *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L} \cap K^{**}(\mathscr{A}))$, *and*

(3) $QF|_{\mathscr{L}} : \mathscr{L} \to D(\mathscr{B})$ *vanishes on the acyclic complexes.*

Then both F *and* $F|_{K^{**}(\mathscr{A})}$ *have left derived functors* $(\boldsymbol{L}^*F, \xi)$ *and* $(\boldsymbol{L}^{**}(F|_{K^{**}(\mathscr{A})}), \zeta)$, *respectively, and the canonical homomorphism*

$$\varphi : \boldsymbol{L}^*F|_{D^{**}(\mathscr{A})} \to \boldsymbol{L}^{**}(F|_{K^{**}(\mathscr{A})})$$

is an isomorphism.

Proposition 14.12 (Dual of Proposition 13.12). *Let* $K^*(\mathscr{A})$ *be a localizing subcategory of* $K(\mathscr{A})$ *and* $F : K^*(\mathscr{A}) \to K(\mathscr{B})$ *a* ∂*-functor. Let* $K^\dagger(\mathscr{B})$ *be a localizing subcategory of* $K(\mathscr{B})$ *and* $G : K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ *a* ∂*-functor. Assume*

(1) $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *for which the hypotheses (1), (2) of Proposition 14.5 are satisfied,*

(2) $K^\dagger(\mathscr{B})$ *has a full triangulated subcategory* $\mathscr{M}$ *for which the hypotheses (1), (2) of Proposition 14.5 are satisfied, and*

(3) $F(K^*(\mathscr{A})) \subset K^\dagger(\mathscr{B})$ *and* $F(\mathscr{L}) \subset \mathscr{M}$.

Then F, G *and* GF *have left derived functors* $(\boldsymbol{L}^*F, \xi)$, $(\boldsymbol{L}^\dagger F, \zeta)$ *and* $(\boldsymbol{L}^*(GF), \psi)$, *respectively,* $\boldsymbol{L}^*F(D^*(\mathscr{A})) \subset D^\dagger(\mathscr{B})$, *and the canonical homomorphism*

$$\varphi : \boldsymbol{L}^\dagger G \circ \boldsymbol{L}^*F \to \boldsymbol{L}^*(GF)$$

is an isomorphism.

Proposition 14.13 (Dual of Proposition 13.13). *Let* $\mathscr{A}'$ *be a thick subcategory of* $\mathscr{A}$ *and* $J : D^-(\mathscr{A}') \to D^-(\mathscr{A})$ *the canonical functor. Let* $F : K^-(\mathscr{A}) \to K(\mathscr{B})$ *be a* ∂*-functor. Assume* $\mathscr{A}$ *has enough injectives and* $\mathscr{A}'$ *has enough* $\mathscr{A}$*-injectives. Then both* F *and* $F|_{K^-(\mathscr{A}')}$ *have left derived functors* $(\boldsymbol{L}^-F, \xi)$ *and* $(\zeta, \boldsymbol{L}^-(F|_{K^-(\mathscr{A}')}))$, *respectively, and the canonical homomorphism*

$$\varphi : \boldsymbol{L}^{-}F \circ J \to \boldsymbol{L}^{-}(F\,|_{K^{-}(\mathscr{A}')})$$

is an isomorphism.

Proposition 14.14 (Dual of Proposition 13.14). *Assume $\mathscr{A}$ has enough projectives. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor which has an exact right adjoint $U : \mathscr{B} \to \mathscr{A}$. Then the following hold.*

(1) *The extended ∂-functor $F : K^{-}(\mathscr{A}) \to K(\mathscr{B})$ has a left derived functor $(\boldsymbol{L}^{-}F, \xi)$ which satisfies $\boldsymbol{L}^{-}F(D^{-}(\mathscr{A})) \subset D^{-}(\mathscr{B})$.*

(2) *For the extended ∂-functor $U : K^{-}(\mathscr{B}) \to K(\mathscr{A})$, there exists $\overline{U} : D^{-}(\mathscr{B}) \to D(\mathscr{A})$ such that $QU = \overline{U}Q$.*

(3) *$\overline{U} : D^{-}(\mathscr{B}) \to D^{-}(\mathscr{A})$ is a right adjoint of $\boldsymbol{L}^{-}F : D^{-}(\mathscr{A}) \to D^{-}(\mathscr{B})$.*

(4) *If F is fully faithful, so is $\boldsymbol{L}^{-}F$.*

§15. Double complexes

Throughout this section, $\mathscr{A}$ is an abelian category. Unless otherwise stated, functors are covariant functors.

Definition 15.1. We denote by $\mathscr{A}^{\mathbb{Z}^2}$ the category of $\mathbb{Z}^2$-graded objects in $\mathscr{A}$, i.e., an object of $\mathscr{A}^{\mathbb{Z}^2}$ is a family $C = \{C^{p,q}\}_{p,q \in \mathbb{Z}}$ of $C^{p,q} \in \mathrm{Ob}(\mathscr{A})$ and a morphism $u : \{C^{p,q}\} \to \{D^{p,q}\}$ is a family $u = \{u^{p,q}\}_{p,q \in \mathbb{Z}}$ of $u^{p,q} \in \mathscr{A}(C^{p,q}, D^{p,q})$. We have two kinds of shift functors $T_1, T_2 : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}^2}$ such that

$$T_1(C)^{p,q} = C^{p+1,q} \quad \text{and} \quad T_1(u)^{p,q} = u^{p+1,q},$$
$$T_2(C)^{p,q} = C^{p,q+1} \quad \text{and} \quad T_2(u)^{p,q} = u^{p,q+1}$$

for $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$ and $u \in \mathscr{A}^{\mathbb{Z}^2}(C, D)$, respectively. A differential (d_1, d_2) of $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$ is a pair of $d_1 \in \mathscr{A}^{\mathbb{Z}^2}(C, T_1(C))$ and $d_2 \in \mathscr{A}^{\mathbb{Z}^2}(C, T_2(C))$ such that

$$T_1(d_1) \circ d_1 = 0, \quad T_2(d_2) \circ d_2 = 0 \quad \text{and} \quad T_1(d_2) \circ d_1 + T_2(d_1) \circ d_2 = 0.$$

A double complex $C^{\bullet\bullet} = (C, d_1, d_2)$ in $\mathscr{A}$ is a $\mathbb{Z}^2$-graded object $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$ together with a differential (d_1, d_2). We denote by $C^2(\mathscr{A})$ the category of double complexes in $\mathscr{A}$: a morphism $u \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ is a morphism $u \in \mathscr{A}^{\mathbb{Z}^2}(C, D)$ such that

$$T_1(u) \circ d_1 = d_1 \circ u \quad \text{and} \quad T_2(u) \circ d_2 = d_2 \circ u.$$

Then the shift functors $T_i : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}^2}$ give rise to autofunctors of $C^2(\mathscr{A})$, called the translations, such that

$$T_i(C^{\bullet\bullet}) = (T_i(C), -T_i(d_1), -T_i(d_2)) \ \ (i = 1, 2)$$

for $C^{\bullet\bullet} = (C, d_1, d_2)$.

Remark 15.1. (1) $T_1 \circ T_2 = T_2 \circ T_1$.

(2) $d_i \in C^2(\mathscr{A})(C^{\bullet\bullet}, T_i(C^{\bullet\bullet}))$ $(i = 1, 2)$ for all $C^{\bullet\bullet} = (C, d_1, d_2) \in \mathrm{Ob}(C^2(\mathscr{A}))$.

Lemma 15.1. *For a triple* (C, d_1, d_2) *of* $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$, $d_1 \in \mathscr{A}^{\mathbb{Z}^2}(C, T_1(C))$ *and* $d_2 \in \mathscr{A}^{\mathbb{Z}^2}(C, T_2(C))$ *the following are equivalent*.

(1) $C^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$.

(2) (a) $C^{p,\bullet} = (\{C^{p,q}\}, \{(-1)^p d_2^{p,q}\}) \in \mathrm{Ob}(C(\mathscr{A}))$ *for all* $p \in \mathbb{Z}$,

(b) $d_1^{p,\bullet} \in C(\mathscr{A})(C^{p,\bullet},\ C^{p+1,\bullet})$ *for all* $p \in \mathbb{Z}$, *and*

(c) $d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ *for all* $p \in \mathbb{Z}$.

(3) (a) $C^{\bullet,q} = (\{C^{p,q}\}, \{(-1)^q d_1^{p,q}\}) \in \mathrm{Ob}(C(\mathscr{A}))$ *for all* $q \in \mathbb{Z}$,

(b) $d_2^{\bullet,q} \in C(\mathscr{A})(C^{\bullet,q},\ C^{\bullet,q+1})$ *for all* $q \in \mathbb{Z}$, *and*

(c) $d_2^{\bullet,q+1} \circ d_2^{\bullet,q} = 0$ *for all* $q \in \mathbb{Z}$.

Proof. Straightforward.

Definition 15.2. Let $C^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$. Then for each $p \in \mathbb{Z}$ the complex

$$C^{p,\bullet} = (\{C^{p,q}\}, \{(-1)^p d_2^{p,q}\})$$

is called the p^{th} row of $C^{\bullet\bullet}$, and for each $q \in \mathbb{Z}$ the complex

$$C^{\bullet,q} = (\{C^{p,q}\}, \{(-1)^q d_1^{p,q}\})$$

is called the q^{th} column of $C^{\bullet\bullet}$.

Lemma 15.2. *Let* $C^{\bullet\bullet},\ D^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$. *Then for* $u \in \mathscr{A}^{\mathbb{Z}^2}(C, D)$ *the following are equivalent*.

(1) $u \in C^2(\mathscr{A})(C^{\bullet\bullet},\ D^{\bullet\bullet})$.

(2) (a) $u^{p,\bullet} = \{u^{p,q}\} \in C(\mathscr{A})(C^{p,\bullet},\ D^{p,\bullet})$ *for all* $p \in \mathbb{Z}$, *and*

(b) $\{u^{p,\bullet}\} \in C(C(\mathscr{A}))(\{C^{p,\bullet}\}, \{D^{p,\bullet}\})$.

(3) (a) $u^{\bullet,q} = \{u^{p,q}\} \in C(\mathscr{A})(C^{\bullet,q},\ D^{\bullet,q})$ *for all* $q \in \mathbb{Z}$, *and*

(b) $\{u^{\bullet,q}\} \in C(C(\mathscr{A}))(\{C^{\bullet,q}\}, \{D^{\bullet,q}\})$.

Proof. Straightforward.

Proposition 15.3. *We have isomorphisms of categories*

$$C^2(\mathscr{A}) \xrightarrow{\sim} C(C(\mathscr{A})),\ C^{\bullet\bullet} \mapsto (\{C^{p,\bullet}\}, \{d_1^{p,\bullet}\}),$$
$$C^2(\mathscr{A}) \xrightarrow{\sim} C(C(\mathscr{A})),\ C^{\bullet\bullet} \mapsto (\{C^{\bullet,q}\}, \{d_2^{\bullet,q}\}).$$

Proof. By Lemmas 15.1 and 15.2.

Definition 15.3. Let $u,\ v \in C^2(\mathscr{A})(C^{\bullet\bullet},\ D^{\bullet\bullet})$. A homotopy $(h_1, h_2) : u \simeq v$ is a pair of morphisms $h_1 \in \mathscr{A}^{\mathbb{Z}^2}(T_1(C), D)$ and $h_2 \in \mathscr{A}^{\mathbb{Z}^2}(T_2(C), D)$ such that

$$h_1 \circ T_2^{-1}(T_1(d_2)) + T_2^{-1}(d_2 \circ h_1) = 0,$$

$$h_2 \circ T_1^{-1}(T_2(d_1)) + T_1^{-1}(d_1 \circ h_2) = 0,$$

$$h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1) + h_2 \circ d_2 + T_2^{-1}(d_2 \circ h_2) = u - v.$$

If there exists a homotopy $(h_1, h_2) : u \simeq v$, we say that u is homotopic to v.

Remark 15.3. Let $C^{\bullet\bullet}$, $D^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$. Then the following hold.

(1) For $h_1 \in \mathscr{A}^{\mathbb{Z}^2}(T_1(C), D)$ the following are equivalent.

(a) $h_1 \circ T_2^{-1}(T_1(d_2)) + T_2^{-1}(d_2 \circ h_1) = 0$.

(b) $h_1^{p,\bullet} = \{h_1^{p,q}\} \in C(\mathscr{A})(C^{p+1,\bullet}, D^{p,\bullet})$ for all $p \in \mathbb{Z}$.

(2) For $h_2 \in \mathscr{A}^{\mathbb{Z}^2}(T_2(C), D)$ the following are equivalent.

(a) $h_2 \circ T_1^{-1}(T_2(d_1)) + T_1^{-1}(d_1 \circ h_2) = 0$.

(b) $h_2^{\bullet,q} = \{h_2^{p,q}\} \in C(\mathscr{A})(C^{\bullet,q+1}, D^{\bullet,q})$ for all $q \in \mathbb{Z}$.

Definition 15.4. In case $\mathscr{A}$ satisfies the condition Ab3*, we define a left exact functor $t : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}}$ such that

$$t(C)^n = \prod_{p+q=n} C^{p,q}$$

for all $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$ and $n \in \mathbb{Z}$, and in case $\mathscr{A}$ satisfies the condition Ab3, we define a right exact functor $t' : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}}$ such that

$$t'(C)^n = \bigoplus_{p+q=n} C^{p,q}$$

for all $C \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}^2})$ and $n \in \mathbb{Z}$. Furthermore, in case $\mathscr{A}$ satisfies the conditions Ab3 and Ab3*, we define a functor $t'' : \mathscr{A}^{\mathbb{Z}^2} \to \mathscr{A}^{\mathbb{Z}}$ by the canonical exact sequence

$$t' \to t \to t'' \to 0.$$

Throughout the rest of this section, $\mathscr{A}$ is assumed to satisfy the condition Ab3*. In the following statements, t can be replaced by t' (in that case, $\mathscr{A}$ is of course assumed to satisfy the condition Ab3 instead of Ab3*).

Lemma 15.4. *The following hold.*

(1) $t \circ T_i = T \circ t$ *for* $i = 1, 2$.

(2) $d = t(d_1) + t(d_2)$ *is a differential of* $t(C)$ *for any* $C^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$, *so that we have a left exact functor*

$$t : C^2(\mathscr{A}) \to C(\mathscr{A}),\ C^{\bullet\bullet} \mapsto (t(C), t(d_1) + t(d_2)).$$

Proof. (1) Obvious.

(2) By the part (1) we have

$$\begin{aligned} T(d) \circ d &= T(t(d_1) + t(d_2)) \circ (t(d_1) + t(d_2)) \\ &= T(t(d_1)) \circ t(d_1) + \{T(t(d_1)) \circ t(d_2) + T(t(d_2)) \circ t(d_1)\} + T(t(d_2)) \circ t(d_2) \\ &= t(T_1(d_1) \circ d_1) + t(T_2(d_1) \circ d_2 + T_1(d_2) \circ d_1) + t(T_2(d_2) \circ d_2) \\ &= 0. \end{aligned}$$

Lemma 15.5. *Let $u, v \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ with $(h_1, h_2) : u \simeq v$ and put $h = t(h_1) + t(h_2)$. Then $h : t(u) \simeq t(v)$.*

Proof. We have

$$\begin{aligned} t(u) - t(v) &= t(u - v) \\ &= t(h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1) + h_2 \circ d_2 + T_2^{-1}(d_2 \circ h_2)) \\ &\quad + t(h_1 \circ T_2^{-1}(T_1(d_2)) + T_2^{-1}(d_2 \circ h_1)) + t(h_2 \circ T_1^{-1}(T_2(d_1)) + T_1^{-1}(d_1 \circ h_2)) \\ &= \{t(h_1) \circ t(d_1) + T^{-1}(t(d_1) \circ t(h_1)) + t(h_2) \circ t(d_2) + T^{-1}(t(d_2) \circ t(h_2))\} \\ &\quad + \{t(h_1) \circ t(d_2) + T^{-1}(t(d_2) \circ t(h_1))\} + \{t(h_2) \circ t(d_1) + T^{-1}(t(d_1) \circ t(h_2))\} \\ &= (t(h_1) + t(h_2)) \circ (t(d_1) + t(d_2)) + T^{-1}(t(d_1) + t(d_2)) \circ T^{-1}(t(h_1) + t(h_2)) \\ &= h \circ d + T^{-1}(d) \circ T^{-1}(h). \end{aligned}$$

Definition 15.5. Each complex $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ is considered as a double complex concentrated in the 0^{th} column, i.e., a double complex such that

$$(X^\bullet)^{p,q} = \begin{cases} X^p & (q = 0) \\ 0 & (q \neq 0) \end{cases}$$

and we get a full embedding $C(\mathscr{A}) \to C^2(\mathscr{A})$.

Remark 15.4. The embedding $C(\mathscr{A}) \to C^2(\mathscr{A})$ preserves homotopy classes, i.e., if $u, v \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ with $h : u \simeq v$, then $(h, 0) : u \simeq v$ for $u, v \in C^2(\mathscr{A})(X^\bullet, Y^\bullet)$.

Definition 15.6. A right resolution of a complex $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ is a morphism $\mu : X^\bullet \to C^{\bullet\bullet}$ in $C^2(\mathscr{A})$ such that $(\{C^{\bullet,q}\}, \{d_2^{\bullet,q}\})$ is a right resolution of $X^\bullet$ in $C(C(\mathscr{A}))$, i.e., $C^{\bullet,q} = 0$ for $q < 0$ and

$$0 \to X^{\bullet} \overset{\mu}{\to} C^{\bullet,0} \to C^{\bullet,1} \to \cdots$$

is an exact sequence in $C(\mathscr{A})$. A right resolution $\mu : X^{\bullet} \to C^{\bullet\bullet}$ of $X^{\bullet}$ is called an injective resolution if every $C^{\bullet,q}$ is an injective object of $C(\mathscr{A})$, and is called a right Cartan-Eilenberg resolution if for each $p \in \mathbb{Z}$

$$0 \to H^p(X^{\bullet}) \to H_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to H_{\mathrm{I}}^{p,1}(C^{\bullet\bullet}) \to \cdots,$$
$$0 \to B^p(X^{\bullet}) \to B_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to B_{\mathrm{I}}^{p,1}(C^{\bullet\bullet}) \to \cdots$$

are injective resolutions of $H^p(X^{\bullet})$ and $B^p(X^{\bullet})$, respectively.

Definition 15.7. For each $p \in \mathbb{Z}$ we define additive functors $Z_{\mathrm{I}}^{p,\bullet}$, $B_{\mathrm{I}}^{p,\bullet}$, $Z_{\mathrm{I}}'^{p,\bullet}$, $B_{\mathrm{I}}'^{p,\bullet}$ and $H_{\mathrm{I}}^{p,\bullet} : C^2(\mathscr{A}) \to C(\mathscr{A})$ as follows

$$Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) = \operatorname{Ker} d_1^{p,\bullet},$$
$$B_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) = \operatorname{Im} d_1^{p-1,\bullet},$$
$$Z_{\mathrm{I}}'^{p,\bullet}(C^{\bullet\bullet}) = \operatorname{Cok} d_1^{p-1,\bullet},$$
$$B_{\mathrm{I}}'^{p,\bullet}(C^{\bullet\bullet}) = \operatorname{Coim} d_1^{p,\bullet} = \operatorname{Im} d_1^{p,\bullet} = B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet}),$$
$$H_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) = Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet})/B_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}).$$

Also, for each $q \in \mathbb{Z}$ we define additive functors $Z_{\mathrm{II}}^{\bullet,q}$, $B_{\mathrm{II}}^{\bullet,q}$, $Z_{\mathrm{II}}'^{\bullet,q}$, $B_{\mathrm{II}}'^{\bullet,q}$ and $H_{\mathrm{II}}^{\bullet,q} : C^2(\mathscr{A}) \to C(\mathscr{A})$ as follows

$$Z_{\mathrm{II}}^{\bullet,q}(C^{\bullet\bullet}) = \operatorname{Ker} d_2^{\bullet,q},$$
$$B_{\mathrm{II}}^{\bullet,q}(C^{\bullet\bullet}) = \operatorname{Im} d_2^{\bullet,q-1},$$
$$Z_{\mathrm{II}}'^{\bullet,q}(C^{\bullet\bullet}) = \operatorname{Cok} d_2^{\bullet,q-1},$$
$$B_{\mathrm{II}}'^{\bullet,q}(C^{\bullet\bullet}) = \operatorname{Coim} d_2^{\bullet,q} = \operatorname{Im} d_2^{\bullet,q} = B_{\mathrm{II}}^{\bullet,q+1}(C^{\bullet\bullet}),$$
$$H_{\mathrm{II}}^{\bullet,q}(C^{\bullet\bullet}) = Z_{\mathrm{II}}^{\bullet,q}(C^{\bullet\bullet})/B_{\mathrm{II}}^{\bullet,q}(C^{\bullet\bullet}).$$

Lemma 15.6. *Let $X^{\bullet} \to C^{\bullet\bullet}$ be a right Cartan-Eilenberg resolution of $X^{\bullet} \in$* Ob($C(\mathscr{A})$). *Then for each $p \in \mathbb{Z}$ we have injective resolutions*

$$0 \to Z^p(X^{\bullet}) \to Z_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}^{p,1}(C^{\bullet\bullet}) \to \cdots,$$
$$0 \to Z'^p(X^{\bullet}) \to Z_{\mathrm{I}}'^{p,0}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}'^{p,1}(C^{\bullet\bullet}) \to \cdots,$$
$$0 \to X^p \to C^{p,0} \to C^{p,1} \to \cdots$$

of $Z^p(X^\bullet)$, $Z'^p(X^\bullet)$ *and* X^p, *respectively.*

Proof. Let $n \in \mathbb{Z}$. Since we have exact sequences

$$0 \to B_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) \to H_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) \to 0,$$
$$0 \to H_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}'^{p,\bullet}(C^{\bullet\bullet}) \to B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet}) \to 0,$$
$$0 \to Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) \to C^{p,\bullet} \to B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet}) \to 0$$

in $C(\mathscr{A})$, by Proposition 1.3 the assertion follows.

Lemma 15.7. *Assume* $\mathscr{A}$ *has enough injectives. Then every* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *has a right Cartan-Eilenberg resolution* $X^\bullet \to C^{\bullet\bullet}$.

Proof. Take injective resolutions $H^p(X^\bullet) \to H^{p,\bullet}$, $B^p(X^\bullet) \to B^{p,\bullet}$ of $H^p(X^\bullet)$ and $B^p(X^\bullet)$, respectively, for all $p \in \mathbb{Z}$. Then, for each $p \in \mathbb{Z}$, by Proposition 2.7 we get an injective resolution $Z^p(X^\bullet) \to Z^{p,\bullet}$ of $Z^p(X^\bullet)$ and a commutative diagram in $C(\mathscr{A})$ with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \to & B^p(X^\bullet) & \to & Z^p(X^\bullet) & \to & H^p(X^\bullet) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^{p,\bullet} & \to & Z^{p,\bullet} & \to & H^{p,\bullet} & \to & 0\,, \end{array}$$

thus again by Proposition 2.7 we get an injective resolution $X^p \to Z^p(X^\bullet)$ of X^p and a commutative diagram in $C(\mathscr{A})$ with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \to & Z^p(X^\bullet) & \to & X^p & \to & B^{p+1}(X^\bullet) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z^{p,\bullet} & \to & C^{p,\bullet} & \to & B^{p+1,\bullet} & \to & 0\,. \end{array}$$

Consequently, we get a right Cartan-Eilenberg resolution $X^\bullet \to C^{\bullet\bullet}$.

Definition 15.8. Let $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ and $X^\bullet \to C^{\bullet\bullet}$, $Y^\bullet \to D^{\bullet\bullet}$ right resolutions of $X^\bullet$ and $Y^\bullet$, respectively. Then a morphism $\hat{u} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ is said to be lying over u if $H_{\mathrm{II}}^{\bullet,0}(\hat{u}) = u$.

Remark 15.6. If $X^\bullet \to C^{\bullet\bullet}$ is a right resolution of $X^\bullet$, then d_1 is lying over d_X.

Lemma 15.8. *Let* $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *and* $X^\bullet \to C^{\bullet\bullet}$, $Y^\bullet \to D^{\bullet\bullet}$ *right Cartan-Eilenberg resolutions of* $X^\bullet$ *and* $Y^\bullet$, *respectively. Then there exists* $\hat{u} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ *lying over* u.

Proof. For each $p \in \mathbb{Z}$, by Lemma 1.8 there exist $\hat{z}^{p,\bullet} \in C(\mathscr{A})(Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}), Z_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}))$ lying over $Z^p(u)$ and $\hat{b}^{p+1,\bullet} \in C(\mathscr{A})(B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet}), B_{\mathrm{I}}^{p+1,\bullet}(D^{\bullet\bullet}))$ lying over $B^{p+1}(u)$ and then by Proposition 3.14 there exists $\hat{u}^{p,\bullet} \in C(\mathscr{A})(C^{p,\bullet}, D^{p,\bullet})$ lying over u^n which makes the following diagram in $C(\mathscr{A})$ commute

$$\begin{array}{ccccccccc} 0 & \to & Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet}) & \to & C^{p,\bullet} & \to & B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet}) & \to & 0 \\ & & \hat{z}^{p,\bullet}\downarrow & & \downarrow \hat{u}^{p,\bullet} & & \downarrow \hat{b}^{p+1,\bullet} & & \\ 0 & \to & Z_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}) & \to & D^{p,\bullet} & \to & B_{\mathrm{I}}^{p+1,\bullet}(D^{\bullet\bullet}) & \to & 0\,. \end{array}$$

It follows by Lemma 15.2 that $\hat{u} = \{u^{p,q}\} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$. Also, $H_{\mathrm{II}}^{p,0}(\hat{u}) = H^0(\hat{u}^{p,\bullet}) = u^p$ for all $p \in \mathbb{Z}$.

Lemma 15.9. *Let* $u, v \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *with* $u \simeq v$. *Let* $\mu : X^\bullet \to C^{\bullet\bullet}$, $\nu : Y^\bullet \to D^{\bullet\bullet}$ *be right Cartan-Eilenberg resolutions of* $X^\bullet$ *and* $Y^\bullet$, *respectively, and* $\hat{u}, \hat{v} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ *lying over* u *and* v, *respectively. Then* $\hat{u} \simeq \hat{v}$.

Proof. Let $h : u \simeq v$.

Claim 1: There exists $h_1 \in \mathscr{A}^{\mathbb{Z}^2}(T_1(C^{\bullet\bullet}), D^{\bullet\bullet})$ such that $\nu \circ h = h_1 \circ T_1(\mu)$ and $d_2 \circ h_1 + T_2(h_1) \circ T_1(d_2) = 0$.

Proof. For each $p \in \mathbb{Z}$, by Lemma 1.8 there exists $h_1^{p,\bullet} \in C(\mathscr{A})(C^{p+1,\bullet}, D^{p,\bullet})$ lying over $h^p \in \mathscr{A}(X^{p+1}, Y^p)$. Since $\nu^p \circ h^p = h_1^{p,0} \circ \mu^{p+1}$ for all $p \in \mathbb{Z}$, we have $\nu \circ h = h_1 \circ T_1(\mu)$. Also, for any $p, q \in \mathbb{Z}$, since

$$(-1)^p\, d_2^{p,q} \circ h_1^{p,q} = h_1^{p,q+1} \circ (-1)^{p+1}\, d_2^{p+1,q},$$

we have $d_2^{p,q} \circ h_1^{p,q} + h_1^{p,q+1} \circ d_2^{p+1,q} = 0$. Thus $d_2 \circ h_1 + T_2(h_1) \circ T_1(d_2) = 0$.

Claim 2: $h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1) \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ and is lying over $u - v$.

Proof. We have

$$d_1 \circ (h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)) = d_1 \circ h_1 \circ d_1$$

$$= T_1(h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)) \circ d_1,$$

$$\begin{aligned} d_2 \circ (h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)) &= d_2 \circ h_1 \circ d_1 + d_2 \circ T_1^{-1}(d_1) \circ T_1^{-1}(h_1) \\ &= -T_2(h_1) \circ T_1(d_2) \circ d_1 + d_2 \circ T_1^{-1}(d_1) \circ T_1^{-1}(h_1) \\ &= T_2(h_1 \circ d_1) \circ d_2 - T_1^{-1}(T_2(d_1)) \circ T_1^{-1}(d_2) \circ T_1^{-1}(h_1) \\ &= T_2(h_1 \circ d_1) \circ d_2 + T_1^{-1}(T_2(d_1)) \circ T_1^{-1}(T_2(h_1)) \circ d_2 \\ &= T_2(h_1 \circ d_1) \circ d_2 + T_1^{-1}(T_2(d_1 \circ h_1)) \circ d_2 \\ &= T_2(h_1 \circ d_1) \circ d_2 + T_2(T_1^{-1}(d_1 \circ h_1)) \circ d_2 \\ &= T_2(h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)) \circ d_2, \end{aligned}$$

$$\begin{aligned} \nu \circ (u - v) &= \nu \circ (h \circ d_X + T^{-1}(d_Y \circ h)) \\ &= \nu \circ h \circ d_X + \nu \circ T^{-1}(d_Y) \circ T^{-1}(h) \\ &= h_1 \circ T_1(\mu) \circ d_X + T_1^{-1}(d_1) \circ T^{-1}(\nu) \circ T^{-1}(h) \\ &= h_1 \circ d_1 \circ \mu + T_1^{-1}(d_1) \circ T^{-1}(\nu \circ h) \\ &= h_1 \circ d_1 \circ \mu + T_1^{-1}(d_1) \circ T_1^{-1}(h_1 \circ T_1(\mu)) \\ &= h_1 \circ d_1 \circ \mu + T_1^{-1}(d_1) \circ T_1^{-1}(h_1) \circ \mu \\ &= (h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)) \circ \mu. \end{aligned}$$

Claim 3: $\phi = \hat{v} + (h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1))$ is lying over u.

Proof. Since $\hat{v}$ is lying over v, and since $h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1)$ is lying over $u - v$, it follows that $\hat{v} + (h_1 \circ d_1 + T_1^{-1}(d_1 \circ h_1))$ is lying over $v + (u - v) = u$.

Claim 4: Put $\psi = \hat{u} - \phi$. Then there exists $h_2 \in \mathscr{A}^{\mathbb{Z}^2}(T_2(C^{\bullet\bullet}), D^{\bullet\bullet})$ such that $\psi = h_2 \circ d_2 + T_2^{-1}(d_2 \circ h_2)$ and $d_1 \circ h_2 + T_1(h_2) \circ T_2(d_1) = 0$.

Proof. Note that ψ is lying over $u - u = 0$. Thus, for each $p \in \mathbb{Z}$, by Lemma 3.12 there exist $b^{p,\bullet} : B_{\mathrm{I}}^{p,\bullet}(\psi) \simeq 0$ and $h^{p,\bullet} : H_{\mathrm{I}}^{p,\bullet}(\psi) \simeq 0$. Then, for each $p \in \mathbb{Z}$, by Proposition 3.14 there exist $z^{p,\bullet} : Z_{\mathrm{I}}^{p,\bullet}(\psi) \simeq 0$ and $h_2^{p,\bullet} : \psi^{p,\bullet} \simeq 0$ such that the following diagrams commute

$$\begin{array}{ccccccccc} 0 & \to & T_2(B_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet})) & \to & T_2(Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet})) & \to & T_2(H_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet})) & \to & 0 \\ & & b^{p,\bullet}\downarrow & & \downarrow z^{p,\bullet} & & \downarrow h^{p,\bullet} & & \\ 0 & \to & B_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}) & \to & Z_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}) & \to & H_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}) & \to & 0, \end{array}$$

$$0 \to T_2(Z_{\mathrm{I}}^{p,\bullet}(C^{\bullet\bullet})) \to T_2(C^{p,\bullet}) \to T_2(B_{\mathrm{I}}^{p+1,\bullet}(C^{\bullet\bullet})) \to 0$$

$$
\begin{array}{ccccccccc}
 & & z^{p,\bullet} \downarrow & & \downarrow h_2^{p,\bullet} & & \downarrow b^{p+1,\bullet} & & \\
0 & \to & Z_{\mathrm{I}}^{p,\bullet}(D^{\bullet\bullet}) & \to & D^{p,\bullet} & \to & B_{\mathrm{I}}^{p+1,\bullet}(D^{\bullet\bullet}) & \to & 0.
\end{array}
$$

It follows that $h_2^{\bullet,q} \in C(\mathscr{A})(C^{\bullet,q+1}, D^{\bullet,q})$ and $\psi^{\bullet,q} = h_2^{\bullet,q} \circ d_2^{\bullet,q} + d_2^{\bullet,q-1} \circ h_2^{\bullet,q-1}$ for all $q \in \mathbb{Z}$. Thus $\psi = h_2 \circ d_2 + T_2^{-1}(d_2 \circ h_2)$. Also, for any $p, q \in \mathbb{Z}$, since

$$(-1)^q d_1^{p,q} \circ h_2^{p,q} = h_2^{p+1,q} \circ (-1)^{q+1} d_1^{p,q+1},$$

$d_1^{p,q} \circ h_2^{p,q} + h_2^{p+1,q} \circ d_1^{p,q+1} = 0$. Thus $d_1 \circ h_2 + T_1(h_2) \circ T_2(d_1) = 0$.

Claim 5: (h_1, h_2): $\hat{u} \simeq \hat{v}$.

Proof. By Claims 1, 3 and 4.

Definition 15.9. For each $n \in \mathbb{Z}$, we define truncation functors $\sigma_{>n}^{\mathrm{II}}, \sigma_{\le n}^{\mathrm{II}} : C^2(\mathscr{A}) \to C^2(\mathscr{A})$ as follows:

$$\sigma_{>n}^{\mathrm{II}}(C^{\bullet\bullet})^{\bullet,q} = \begin{cases} C^{\bullet,q} & (q > n) \\ B_{\mathrm{II}}^{\bullet,n+1}(C^{\bullet\bullet}) & (q = n), \\ 0 & (q < n) \end{cases} \quad \sigma_{\le n}^{\mathrm{II}}(C^{\bullet\bullet})^{\bullet,q} = \begin{cases} 0 & (q > n) \\ Z_{\mathrm{II}}^{\bullet,n}(X^{\bullet}) & (q = n) \\ C^{\bullet,q} & (q < n) \end{cases}$$

for $C^{\bullet\bullet} \in C^2(\mathscr{A})$. We set $\sigma_{\ge n}^{\mathrm{II}} = \sigma_{>n-1}^{\mathrm{II}}$ and $\sigma_{<n}^{\mathrm{II}} = \sigma_{\le n-1}^{\mathrm{II}}$.

Lemma 15.10. *The functors* $\sigma_{>n}^{\mathrm{II}}, \sigma_{\le n}^{\mathrm{II}} : C^2(\mathscr{A}) \to C^2(\mathscr{A})$ *preserve homotopy classes.*

Proof. Straightforward.

Lemma 15.11. *Let* $C^{\bullet\bullet} \in \mathrm{Ob}(C^2(\mathscr{A}))$ *and* $n \in \mathbb{Z}$. *Assume* $H_{\mathrm{II}}^{\bullet,n}(C^{\bullet\bullet}) = 0$ *and put* $Z^{\bullet} = Z_{\mathrm{II}}^{\bullet,n}(C^{\bullet\bullet})$. *Then we have exact sequences in* $C(\mathscr{A})$ *of the form*

$$0 \to t(\sigma_{<n}^{\mathrm{II}}(C^{\bullet\bullet})) \to t(\sigma_{\le n}^{\mathrm{II}}(C^{\bullet\bullet})) \to T^{-n}(C(\mathrm{id}_Z)) \to 0,$$

$$0 \to T^{-n}(C(\mathrm{id}_Z)) \to t(\sigma_{\ge n}^{\mathrm{II}}(C^{\bullet\bullet})) \to t(\sigma_{>n}^{\mathrm{II}}(C^{\bullet\bullet})) \to 0.$$

Proof. Let $\phi : \sigma_{<n}^{\mathrm{II}}(C^{\bullet\bullet}) \to \sigma_{\le n}^{\mathrm{II}}(C^{\bullet\bullet})$ be a canonical monomorphism and $\psi : \sigma_{\ge n}^{\mathrm{II}}(C^{\bullet\bullet}) \to \sigma_{>n}^{\mathrm{II}}(C^{\bullet\bullet})$ a canonical monomorphism. Put $D^{\bullet\bullet} = \mathrm{Cok}\,\phi \cong \mathrm{Ker}\,\psi$. Then, since each double complex has bounded rows, we have exact sequences.

$$0 \to t(\sigma_{<n}^{\mathrm{II}}(C^{\bullet\bullet})) \to t(\sigma_{\le n}^{\mathrm{II}}(C^{\bullet\bullet})) \to t(D^{\bullet\bullet}) \to 0,$$

$$0 \to t(D^{\bullet\bullet}) \to t(\sigma^{\mathrm{II}}_{\geq n}(C^{\bullet\bullet})) \to t(\sigma^{\mathrm{II}}_{>n}(C^{\bullet\bullet})) \to 0.$$

Next, since

$$D^{\bullet,q} = \begin{cases} Z^{\bullet} & (q = n-1, n) \\ 0 & (q \neq n-1, n) \end{cases}$$

and $d_2^{\bullet, n-1} = \mathrm{id}_Z$, we have $t(D^{\bullet\bullet}) \cong T^{-n}(C(\mathrm{id}_Z))$.

Proposition 15.12. *Let* $\mu : X^{\bullet} \to C^{\bullet\bullet}$ *be a right resolution of* $X^{\bullet} \in \mathrm{Ob}(C(\mathscr{A}))$. *Assume there exists* $n_0 \geq 0$ *such that* $C^{\bullet,q} = 0$ *for* $q > n_0$. *Then* $t(\mu) : X^{\bullet} \to t(C^{\bullet\bullet})$ *is a quasi-isomorphism*.

Proof. If $n_0 = 0$, then $X^{\bullet} \cong C^{\bullet\bullet}$. Assume $n_0 \geq 1$. Since $X^{\bullet} \cong \sigma^{\mathrm{II}}_{<1}(C^{\bullet\bullet})$ and $\sigma^{\mathrm{II}}_{\leq n_0}(C^{\bullet\bullet}) \cong C^{\bullet\bullet}$, by Lemma 15.11 the assertion follows.

Definition 15.10. A left resolution of a complex $X^{\bullet} \in \mathrm{Ob}(C(\mathscr{A}))$ is a morphism $\varepsilon : C^{\bullet\bullet} \to X^{\bullet}$ in $C^2(\mathscr{A})$ such that $(\{C^{\bullet,q}\}, \{d_2^{\bullet,q}\})$ is a left resolution of $X^{\bullet}$ in $C(C(\mathscr{A}))$, i.e., $C^{\bullet,q} = 0$ for $q > 0$ and

$$\cdots \to C^{\bullet,-1} \to C^{\bullet,0} \xrightarrow{\varepsilon} X^{\bullet} \to 0$$

is an exact sequence in $C(\mathscr{A})$. A left resolution $\varepsilon : C^{\bullet\bullet} \to X^{\bullet}$ of $X^{\bullet}$ is called a projective resolution if every $C^{\bullet,q}$ is a projective object of $C(\mathscr{A})$, and is called a left Cartan-Eilenberg resolution if for each $p \in \mathbb{Z}$

$$\cdots \to H_{\mathrm{I}}^{p,-1}(C^{\bullet\bullet}) \to H_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to H^p(X^{\bullet}) \to 0,$$
$$\cdots \to B_{\mathrm{I}}^{p,-1}(C^{\bullet\bullet}) \to B_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to B^p(X^{\bullet}) \to 0$$

are projective resolutions of $H^p(X^{\bullet})$ and $B^p(X^{\bullet})$, respectively.

Lemma 15.13 (Dual of Lemma 15.6). *Let* $C^{\bullet\bullet} \to X^{\bullet}$ *be a left Cartan-Eilenberg resolution of* $X^{\bullet} \in \mathrm{Ob}(C(\mathscr{A}))$. *Then for each* $p \in \mathbb{Z}$ *we have projective resolutions*

$$\cdots \to Z_{\mathrm{I}}^{p,-1}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}^{p,0}(C^{\bullet\bullet}) \to Z^p(X^{\bullet}) \to 0,$$
$$\cdots \to Z_{\mathrm{I}}'^{p,-1}(C^{\bullet\bullet}) \to Z_{\mathrm{I}}'^{p,0}(C^{\bullet\bullet}) \to Z'^p(X^{\bullet}) \to 0,$$
$$\cdots \to C^{p,-1} \to C^{p,0} \to X^p \to 0$$

of $Z^p(X^{\bullet})$, $Z'^p(X^{\bullet})$ *and* X^p, *respectively.*

Lemma 15.14 (Dual of Lemma 15.7). *Assume $\mathscr{A}$ has enough projectives. Then every* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *has a left Cartan-Eilenberg resolution* $C^{\bullet\bullet} \to X^\bullet$.

Definition 15.11. Let $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ and $C^{\bullet\bullet} \to X^\bullet$, $D^{\bullet\bullet} \to Y^\bullet$ left resolutions of $X^\bullet$ and $Y^\bullet$, respectively. Then a morphism $\hat{u} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ is said to be lying over u if $H_{\mathrm{II}}^{\bullet,0}(\hat{u}) = u$.

Remark 15.7. If $C^{\bullet\bullet} \to X^\bullet$ is a left resolution of $X^\bullet$, then d_1 is lying over d_X.

Lemma 15.15 (Dual of Lemma 15.8). *Let* $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *and let* $C^{\bullet\bullet} \to X^\bullet$, $D^{\bullet\bullet} \to Y^\bullet$ *be left Cartan-Eilenberg resolutions of* $X^\bullet$ *and* $Y^\bullet$, *respectively. Then there exists* $\hat{u} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ *lying over* u.

Lemma 15.16 (Dual of Lemma 15.9). *Let* $u, v \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *with* $u \simeq v$. *Let* $C^{\bullet\bullet} \to X^\bullet$, $D^{\bullet\bullet} \to Y^\bullet$ *be left Cartan-Eilenberg resolutions of* $X^\bullet$ *and* $Y^\bullet$, *respectively, and* $\hat{u}, \hat{v} \in C^2(\mathscr{A})(C^{\bullet\bullet}, D^{\bullet\bullet})$ *lying over u and v, respectively. Then* $\hat{u} \simeq \hat{v}$.

Proposition 15.17 (Dual of Proposition 15.12). *Let* $\varepsilon : C^{\bullet\bullet} \to X^\bullet$ *be a left resolution of* $X^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. *Assume there exists* $n_0 \leq 0$ *such that* $C^{\bullet,q} = 0$ *for all* $q < n_0$. *Then* $t(\varepsilon) : t(C^{\bullet\bullet}) \to X^\bullet$ *is a quasi-isomorphism.*

§16. Left exact functors of finite cohomological dimension

Throughout this section, $\mathscr{A}$ and $\mathscr{B}$ are abelian categories. Unless otherwise stated, functors are covariant functors.

Proposition 16.1. *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an exact functor. Then the following hold*

(1) *There exists a unique* ∂*-functor* $F' : D(\mathscr{A}) \to D(\mathscr{B})$ *such that* $QF = F'Q$.

(2) (id_{QF}, F') *is a right derived functor of* $F : K(\mathscr{A}) \to K(\mathscr{B})$.

(3) (F', id_{QF}) *is a left derived functor of* $F : K(\mathscr{A}) \to K(\mathscr{B})$.

(4) $\mathrm{Ker}\, F' = D_{\mathscr{A}'}(\mathscr{A})$, *where* $\mathscr{A}' = \mathrm{Ker}\, F$.

Proof. Since $QF : K(\mathscr{A}) \to D(\mathscr{B})$ vanishes on the acyclic complexes, (1), (2) and (3) follow by Propositions 13.1 and 14.1.

(4) Let $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$. Since $Q(F(X^\bullet)) = 0$ if and only if $F(X^\bullet)$ is acyclic, it follows that $F'(Q(X^\bullet)) = 0$ if and only if $F(H^n(X^\bullet)) = 0$ for all $n \in \mathbb{Z}$.

Lemma 16.2. *Let* $\mathscr{I}$ *be a subcollection of* $\mathrm{Ob}(\mathscr{A})$ *closed under finite direct sums. Assume*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $X \to I$ *in* $\mathscr{A}$ *with* $I \in \mathscr{I}$, *and*

(2) *there exists an integer* $n \geq 1$ *such that if*

$$X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^1, \cdots, X^{n-1} \in \mathscr{I}$ *then* $X^n \in \mathscr{I}$.

Then for any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *there exists a monomorphism* $X^\bullet \to I^\bullet$ *in* $C(\mathscr{A})$ *with* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$ *which is a quasi-isomorphism.*

Proof. Let $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$. By hypothesis (1) and Lemma 1.7 we have a right resolution

$$0 \to X^\bullet \xrightarrow{\mu} I^{\bullet,0} \to \cdots \to I^{\bullet,n-1} \to I^{\bullet,n} \to 0$$

of $X^\bullet$ with $I^{\bullet,0}, \cdots, I^{\bullet,n-1} \in \mathrm{Ob}(K(\mathscr{I}))$. Then by hypothesis (2) $I^{\bullet,n} \in \mathrm{Ob}(K(\mathscr{I}))$ and $t(I^{\bullet\bullet}) \in \mathrm{Ob}(K(\mathscr{I}))$. Also, by Proposition 15.12 $t(\mu) : X^\bullet \to t(I^{\bullet\bullet})$ is a quasi-isomorphism.

Lemma 16.3. *Let* $F : \mathscr{A} \to \mathscr{B}$ *be a left exact functor. Assume there exists a subcollection* $\mathscr{I}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *there exists an integer* $n \geq 1$ *such that if*

$$X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^1, \cdots, X^{n-1} \in \mathscr{I}$ *then* $X^n \in \mathscr{I}$, *and*

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{I}$, *then the induced sequence* $0 \to FX \to FY \to FZ \to 0$ *is exact.*

Then $QF : K(\mathscr{I}) \to D(\mathscr{B})$ *vanishes on the acyclic complexes.*

Proof. Let $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$ be an acyclic complex. Note that by hypothesis (1) $Z^i(I^\bullet) \in \mathscr{I}$ for all $i \in \mathbb{Z}$ and that, since F is left exact, $F(Z^i(I^\bullet)) \cong Z^i(F(I^\bullet))$ for all $i \in \mathbb{Z}$. Let $i \in \mathbb{Z}$. For an exact sequence

$$0 \to Z^i(I^\bullet) \to I^i \to Z^{i+1}(I^\bullet) \to 0,$$

since $Z^i(I^\bullet), I^i, Z^{i+1}(I^\bullet) \in \mathscr{I}$, by hypothesis (2) the induced sequence

$$0 \to F(Z^i(I^\bullet)) \to F(I^i) \to F(Z^{i+1}(I^\bullet)) \to 0$$

is exact. Thus the canonical sequence

$$0 \to Z^i(F(I^\bullet)) \to F(I^\bullet)^i \to Z^{i+1}(F(I^\bullet)) \to 0$$

is exact. It follows that $F(I^\bullet)$ is acyclic and $Q(F(I^\bullet)) = 0$.

Proposition 16.4. *Let* $F : \mathscr{A} \to \mathscr{B}$ *be a left exact functor. Assume there exists a subcollection* $\mathscr{I}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $X \to I$ *in* $\mathscr{A}$ *with* $I \in \mathscr{I}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X \in \mathscr{I}$, *then* $Y \in \mathscr{I}$ *if and only if* $Z \in \mathscr{I}$,

(3) *there exists an integer* $n \geq 1$ *such that if*

$$X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^1, \cdots, X^{n-1} \in \mathscr{I}$ *then* $X^n \in \mathscr{I}$, *and*

(4) *for any exact sequece* $0 \to X \to Y \to Z \to 0$ *in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{I}$, *the induced sequence* $0 \to FX \to FY \to FZ \to 0$ *is exact.*

Then both $(\xi, \boldsymbol{R}F)$ *and* $(\zeta, \boldsymbol{R}^+F)$ *exist and the canonical homomorphism*

$$\varphi : \boldsymbol{R}^+F \to \boldsymbol{R}F\,|_{D^+(\mathscr{A})}$$

is an isomorphism. Furthermore, ξ_I *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$.

Proof. According to Proposition 13.11, it suffices to show the following.

Claim : (1) $K(\mathscr{I})$ is a full triangulated subcategory of $K(\mathscr{A})$.

(2) For any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$.

(3) For any $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$, there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}) \cap K^+(\mathscr{A}))$.

(4) $QF : K(\mathscr{I}) \to D(\mathscr{B})$ vanishes on the acyclic complexes.

Proof. (1) By hypothesis (2) and Proposition 6.1(2).

(2) By hypotheses (1), (3) and Lemma 16.2.

(3) By hypothesis (1) and Proposition 4.7.

(4) By hypotheses (3), (4) and Lemma 16.3.

Remark 16.1. In Proposition 16.4, F has cohomological dimension $\leq n$ on $\mathscr{A}$, i.e., $\boldsymbol{R}^i F$ vanishes on $\mathscr{A}$ for $i > n$.

Definition 16.1. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Assume the extended ∂-functor $F : K^+(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $\boldsymbol{R}^+F : D^+(\mathscr{A}) \to D(\mathscr{B})$. Then an object $X \in \mathrm{Ob}(\mathscr{A})$ is called F-acyclic if $\boldsymbol{R}^i F(X) = 0$ for $i > 0$.

Corollary 16.5. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Assume there exists a subcollection $\mathscr{I}$ of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$,*

(2) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, then $Y \in \mathscr{I}$ if and only if $Z \in \mathscr{I}$,*

(3) *for any exact sequece $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ with $X, Y, Z \in \mathscr{I}$, the induced sequence $0 \to FX \to FY \to FZ \to 0$ is exact, and*

(4) *F has finite cohomological dimension on $\mathscr{A}$, i.e., there exists $n \geq 1$ such that $\boldsymbol{R}^i F$ vanishes on $\mathscr{A}$ for $i > n$ (Note that by Corollary 13.7 $\boldsymbol{R}^+F$ exists).*

Then $(\xi, \boldsymbol{R}F)$ exists and the canonical homomorphism

$$\varphi : \boldsymbol{R}^+F \to \boldsymbol{R}F\,|_{D^+(\mathscr{A})}$$

is an isomorphism. Furthermore, ξ_I is an isomorphism for all $I^\bullet \in \mathrm{Ob}(K(\hat{\mathscr{I}}))$, where $\hat{\mathscr{I}}$ is the collection of F-acyclic objects $X \in \mathrm{Ob}(\mathscr{A})$.

Proof. According to Proposition 16.4 it suffices to show the following.

Claim : (1) For any $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $X \to I$ in $\mathcal{A}$ with $I \in \hat{\mathcal{I}}$.

(2) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$ with $X \in \hat{\mathcal{I}}$, then $Y \in \hat{\mathcal{I}}$ if and only if $Z \in \hat{\mathcal{I}}$.

(3) If $X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$ is an exact sequence in $\mathcal{A}$ with $X^0, X^1, \cdots, X^{n-1} \in \hat{\mathcal{I}}$, then $X^n \in \hat{\mathcal{I}}$.

(4) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathcal{A}$ with $X \in \hat{\mathcal{I}}$, then the induced sequence $0 \to FX \to FY \to FZ \to 0$ is exact.

Proof. (1) It suffices to show $\mathcal{I} \subset \hat{\mathcal{I}}$. For any $X \in \mathcal{I}$ and $i > 0$, by Corollary 13.7 $\boldsymbol{R}^iF(X) = H^i(\boldsymbol{R}^+F(X)) \cong H^i(Q(F(X))) = 0$.

(2) Since we have a long exact sequence

$$\cdots \to \boldsymbol{R}^iF(X) \to \boldsymbol{R}^iF(Y) \to \boldsymbol{R}^iF(Z) \to \boldsymbol{R}^{i+1}F(X) \to \cdots,$$

$\boldsymbol{R}^iF(Y) \cong \boldsymbol{R}^iF(Z)$ for all $i > 0$.

(3) Put $Z^i = \mathrm{Ker}(X^i \to X^{i+1})$ for $0 \leq i < n$. Then by Proposition 13.4

$$\boldsymbol{R}^jF(X^n) \cong \boldsymbol{R}^{j+1}F(Z^{n-1}) \cong \cdots \cong \boldsymbol{R}^{j+n}F(Z^0)$$

for all $j > 0$. It follows by hypothesis (4) that $X^n \in \hat{\mathcal{I}}$.

(4) Since F is left exact, $F \xrightarrow{\sim} \boldsymbol{R}^0F$ canonically. Thus, since $\boldsymbol{R}^1F(X) = 0$, by Proposition 13.4 the induced sequence $0 \to FX \to FY \to FZ \to 0$ is exact.

Lemma 16.6 (Dual of Lemma 16.2). *Let* $\mathcal{P}$ *be a subcollection of* $\mathrm{Ob}(\mathcal{A})$ *closed under finite direct sums. Assume*

(1) *for any* $X \in \mathrm{Ob}(\mathcal{A})$ *there exists a epimorphism* $P \to X$ *in* $\mathcal{A}$ *with* $P \in \mathcal{P}$, *and*

(2) *there exists an integer* $n \geq 1$ *such that if*

$$0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0$$

is an exact sequence in $\mathcal{A}$ *with* $X^0, X^{-1}, \cdots, X^{-n+1} \in \mathcal{P}$ *then* $X^{-n} \in \mathcal{P}$.

Then for any $X^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$ *there exists an epimorphism* $P^\bullet \to X^\bullet$ *in* $C(\mathcal{A})$ *with* $P^\bullet \in \mathrm{Ob}(K(\mathcal{P}))$ *which is a quasi-isomorphism.*

Lemma 16.7 (Dual of Lemma 16.3). *Let* $G : \mathcal{A} \to \mathcal{B}$ *be a right exact functor. Assume there exists a subcollection* $\mathcal{P}$ *of* $\mathrm{Ob}(\mathcal{A})$ *such that*

(1) *there exists an integer* $n \geq 1$ *such that if*

$$0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^{-1}, \cdots, X^{-n+1} \in \mathscr{P}$ *then* $X^{-n} \in \mathscr{P}$, *and*

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact.*

Then $QG : K(\mathscr{P}) \to D(\mathscr{B})$ *vanishes on the acyclic complexes.*

Proposition 16.8 (Dual of Proposition 16.4). *Let* $G : \mathscr{A} \to \mathscr{B}$ *be a right exact functor. Assume there exists a subcollection* $\mathscr{P}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $P \to X$ *in* $\mathscr{A}$ *with* $P \in \mathscr{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $Z \in \mathscr{P}$, *then* $Y \in \mathscr{P}$ *if and only if* $X \in \mathscr{P}$,

(3) *there exists an integer* $n \geq 1$ *such that if*

$$0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^{-1}, \cdots, X^{-n+1} \in \mathscr{P}$ *then* $X^{-n} \in \mathscr{P}$, *and*

(4) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact.*

Then both $(\boldsymbol{L}G, \xi)$ *and* $(\boldsymbol{L}^-G, \zeta)$ *exist and the canonical homomorphism*

$$\varphi : \boldsymbol{L}G\,|_{D^-(\mathscr{A})} \to \boldsymbol{L}^-G$$

is an isomorphism. Furthermore, ξ_P *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(K(\mathscr{P}))$.

Remark 16.2. In Proposition 16.8, G has homological dimension $\leq n$ on $\mathscr{A}$, i.e., $\boldsymbol{L}_iG$ vanishes on $\mathscr{A}$ for $i > n$.

Definition 16.2. Let $G : \mathscr{A} \to \mathscr{B}$ be a right exact functor. Assume the extended ∂-functor $G : K^-(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor $\boldsymbol{L}^-G : D^-(\mathscr{A}) \to D(\mathscr{B})$. Then an object $X \in \mathrm{Ob}(\mathscr{A})$ is called G-acyclic if $\boldsymbol{L}_iG(X) = 0$ for $i > 0$.

Corollary 16.9 (Dual of Corollary 16.5). *Let* $G : \mathscr{A} \to \mathscr{B}$ *be a right exact functor. Assume there exists a subcollection* $\mathscr{P}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $P \to X$ *in* $\mathscr{A}$ *with* $P \in \mathscr{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $Z \in \mathscr{P}$, *then* $Y \in \mathscr{P}$ *if and only if* $X \in \mathscr{P}$,

(3) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact, and*

(4) G *has finite homological dimension on* $\mathscr{A}$, *i.e., there exists* $n \geq 1$ *such that* $\boldsymbol{L}_iG$

vanishes on $\mathscr{A}$ for $i > n$ (Note that by Corollary 14.7 $\boldsymbol{L}^{-}G$ exists).

Then ($\boldsymbol{L}G$, ξ) exists and the canonical homomorphism

$$\varphi : \boldsymbol{L}G\,|_{D^-(\mathscr{A})} \to \boldsymbol{L}^{-}G$$

is an isomorphism. Furthermore, ξ_P is an isomorphism for all $P^\bullet \in \mathrm{Ob}(K(\hat{\mathscr{P}}))$, where $\hat{\mathscr{P}}$ is the collection of G-acyclic objects $X \in \mathrm{Ob}(\mathscr{A})$.

Proposition 16.10. *Assume $\mathscr{A}$ has enough injectives and $\mathscr{B}$ has enough projectives. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor which has a left adjoint $G : \mathscr{B} \to \mathscr{A}$. Assume F has finite cohomological dimension on $\mathscr{A}$ and G has finite homological dimension on $\mathscr{B}$ (Note that by Corollary 13.7 $\boldsymbol{R}^{+}F$ exists and by Corollary 14.7 $\boldsymbol{L}^{-}G$ exists). Then the following hold.*

(1) *The extended ∂-functor $F : K(\mathscr{A}) \to K(\mathscr{B})$ has a right derived functor (ξ, $\boldsymbol{R}F$).*

(2) *The extended ∂-functor $G : K(\mathscr{B}) \to K(\mathscr{A})$ has a left derived functor ($\boldsymbol{L}G$, ζ).*

(3) *$\boldsymbol{L}G$ is a left adjoint of $\boldsymbol{R}F$.*

(4) *Assume G (resp. F) is exact. Then, if F (resp. G) is fully faithful, so is $\boldsymbol{R}F$ (resp. $\boldsymbol{L}G$).*

Proof. (1) By Corollaries 13.7 and 16.5.

(2) By Corollaries 14.7 and 16.9.

(3) By Proposition 3.10 G is a left adjoint of F. Let $\varepsilon : \mathbf{1}_{K(\mathscr{B})} \to FG$, $\delta : GF \to \mathbf{1}_{K(\mathscr{A})}$ be the unit and the counit, respectively.

Claim 1: There exists $\theta \in \mathrm{Hom}\,(\boldsymbol{L}G \circ \boldsymbol{R}F, \mathbf{1}_{D(\mathscr{A})})$ such that $Q\delta \circ \zeta_F = \theta_Q \circ \boldsymbol{L}G\xi$.

Proof. Let $\mathscr{I}$ be the collection of objects $X \in \mathrm{Ob}(\mathscr{A})$ such that $\boldsymbol{R}^iF(X) = 0$ for $i > 0$. By Corollary 16.5 and Lemma 16.2, for any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$. Also, ξ_I is an isomorphism for all $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$. For each $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$, take a quasi-isomorphism $s : X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$ and set

$$\overline{\theta}_X = Q(s)^{-1} \circ Q\delta_I \circ \zeta_{FI} \circ \boldsymbol{L}G(\xi_I^{-1}) \circ \boldsymbol{L}G(\boldsymbol{R}F(Q(s))) : \boldsymbol{L}G(\boldsymbol{R}F(Q(X^\bullet))) \to Q(X^\bullet).$$

Then $\overline{\theta} \in \mathrm{Hom}\,(\boldsymbol{L}G \circ \boldsymbol{R}F \circ Q, Q)$ and $Q\delta \circ \zeta_F = \overline{\theta} \circ \boldsymbol{L}G\xi$. By Proposition 9.11 there exists $\theta \in \mathrm{Hom}\,(\boldsymbol{L}G \circ \boldsymbol{R}F, \mathbf{1}_{D(\mathscr{A})})$ such that $\overline{\theta} = \theta_Q$.

Claim 2: There exists $\eta \in \mathrm{Hom}\,(\mathbf{1}_{D(\mathscr{B})}, \boldsymbol{R}F \circ \boldsymbol{L}G)$ such that $\xi_G \circ Q\varepsilon = \boldsymbol{R}F\zeta \circ \eta_Q$.

Proof. By the dual argument of Claim 1.

Claim 3: $\boldsymbol{R}F\theta \circ \eta_{\boldsymbol{R}F} = \mathrm{id}_{\boldsymbol{R}F}$.

Proof. We have commutative diagrams

$$\begin{array}{ccccc}
Q \circ F & \xrightarrow{Q\varepsilon_F} & Q \circ F \circ G \circ F & \xrightarrow{QF\delta} & Q \circ F \\
\eta_{QF} \downarrow & & \downarrow \xi_{GF} & & \downarrow \xi \\
\boldsymbol{R}F \circ \boldsymbol{L}G \circ Q \circ F & \xrightarrow{\boldsymbol{R}F\zeta_F} & \boldsymbol{R}F \circ Q \circ G \circ F & \xrightarrow{\boldsymbol{R}FQ\delta} & \boldsymbol{R}F \circ Q,
\end{array}$$

$$\begin{array}{ccccc}
Q \circ F & \xrightarrow{\eta_{QF}} & \boldsymbol{R}F \circ \boldsymbol{L}G \circ Q \circ F & \xrightarrow{\boldsymbol{R}F\zeta_F} & \boldsymbol{R}F \circ Q \circ G \circ F \\
\xi \downarrow & & \downarrow \boldsymbol{R}F\boldsymbol{L}G\xi & & \downarrow \boldsymbol{R}FQ\delta \\
\boldsymbol{R}F \circ Q & \xrightarrow{\eta_{\boldsymbol{R}FQ}} & \boldsymbol{R}F \circ \boldsymbol{L}G \circ \boldsymbol{R}F \circ Q & \xrightarrow{\boldsymbol{R}F\theta_Q} & \boldsymbol{R}F \circ Q.
\end{array}$$

Thus $(\boldsymbol{R}F\theta \circ \eta_{\boldsymbol{R}F})_Q \circ \xi = \xi \circ Q(F\delta \circ \varepsilon_F) = \xi$ and by Proposition 13.2 $\boldsymbol{R}F\theta \circ \eta_{\boldsymbol{R}F} = \mathrm{id}_{\boldsymbol{R}F}$.

Claim 4: $\theta_{\boldsymbol{L}G} \circ \boldsymbol{L}G\eta = \mathrm{id}_{\boldsymbol{L}G}$.

Proof. By the dual argument of Claim 3.

(4) Note that ζ is an isomorphism. Assume δ is an isomorphism. Then for any $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$, since $Q\delta \circ \zeta_F = \theta_Q \circ \boldsymbol{L}G\xi$, and since ξ_I is an isomorphism, θ_{QI} is an isomorphism. It follows that θ is an isomorphism.

Proposition 16.11. *Assume $\mathscr{A}$ has enough injectives and let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Assume F has cohomological dimension $\le n$ on $\mathscr{A}$, i.e., $\boldsymbol{R}^iF$ vanishes on $\mathscr{A}$ for $i > n$ (Note that by Corollary 13.7 $\boldsymbol{R}^+F$ exists). Put $G = \boldsymbol{R}^nF : \mathscr{A} \to \mathscr{B}$. Let $\mathscr{P}$ be the collection of $X \in \mathrm{Ob}(\mathscr{A})$ with $\boldsymbol{R}^iF(X) = 0$ for i n and assume that for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists an epimorphism $P \to X$ in $\mathscr{A}$ with $P \in \mathscr{P}$. Then both $\boldsymbol{R}F$ and $\boldsymbol{L}G$ exist and there exists an isomorphism of ∂-functors $\eta : \boldsymbol{R}F \xrightarrow{\sim} \boldsymbol{L}G \circ T^{-n}$.*

Proof. Let $\mathscr{I}$ be the collection of $X \in \mathrm{Ob}(\mathscr{A})$ with $\boldsymbol{R}^iF(X) = 0$ for $i > 0$.

Claim 1: (1) $\mathscr{I}$ contains every injective objects of $\mathscr{A}$, so that for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$.

(2) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, then $Y \in \mathscr{I}$ if and only if $Z \in \mathscr{I}$.

(3) If $X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$ is an exact sequence in $\mathscr{A}$ with $X^0, X^1, \cdots, X^{n-1} \in \mathscr{I}$, then $X^n \in \mathscr{I}$.

(4) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, then the induced

sequence $0 \to FX \to FY \to FZ \to 0$ is exact.

Proof. (1) Let $I \in \mathrm{Ob}(\mathscr{A})$ be injective. Then $\boldsymbol{R}^{+}F(I) \cong F(I)$ and $\boldsymbol{R}^{i}F(X) \cong H^{i}(F(I)) = 0$ for all $i > 0$.

(2) $\boldsymbol{R}^{i}F(Y) \cong \boldsymbol{R}^{i}F(Z)$ for all $i > 0$.

(3) Put $Z^{i} = \mathrm{Ker}(X^{i} \to X^{i+1})$ for $0 \le i < n$. Then $\boldsymbol{R}^{i}F(X^{n}) \cong \boldsymbol{R}^{i+1}F(Z^{n-1}) \cong \cdots \cong \boldsymbol{R}^{i+n}F(Z^{0}) = 0$ for all $i > 0$.

(4) Since F is left exact, $F \xrightarrow{\sim} \boldsymbol{R}^{0}F$ canonically. Also, since $\boldsymbol{R}^{1}F(X) = 0$, the induced sequence $0 \to \boldsymbol{R}^{0}F(X) \to \boldsymbol{R}^{0}F(Y) \to \boldsymbol{R}^{0}F(Z) \to 0$ is exact.

Claim 2: (1) For any $X^{\bullet} \in \mathrm{Ob}(K(\mathscr{A}))$ there exists a quasi-isomorphism $X^{\bullet} \to I^{\bullet}$ with $I^{\bullet} \in \mathrm{Ob}(K(\mathscr{I}))$.

(2) $QF : K(\mathscr{I}) \to D(\mathscr{B})$ vanishes on the acyclic complexes.

Proof. (1) By Claim 1 and Lemma 16.2.

(2) By Claim 1 and Lemma 16.3.

Claim 3: $\boldsymbol{R}F$ exists.

Proof. By Claim 2 and Proposition 13.6.

Claim 4: $G = \boldsymbol{R}^{n}F : \mathscr{A} \to \mathscr{B}$ is right exact.

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathscr{A}$. Then, since $\boldsymbol{R}^{n+1}F(X) = 0$, the induced sequence $\boldsymbol{R}^{n}F(X) \to \boldsymbol{R}^{n}F(Y) \to \boldsymbol{R}^{n}F(Z) \to 0$ is exact.

Claim 5: (1) For any $X \in \mathrm{Ob}(\mathscr{A})$ there exists an epimorphism $P \to X$ in $\mathscr{A}$ with $P \in \mathscr{P}$.

(2) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $Z \in \mathscr{P}$, then $Y \in \mathscr{P}$ if and only if $X \in \mathscr{P}$.

(3) If $0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^{0}$ is an exact sequence in $\mathscr{A}$ with $X^{0}, X^{-1}, \cdots, X^{-n+1} \in \mathscr{P}$, then $X^{-n} \in \mathscr{P}$.

(4) If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $Z \in \mathscr{P}$, then the induced sequence $0 \to GX \to GY \to GZ \to 0$ is exact.

Proof. (1) By hypothesis.

(2) $\boldsymbol{R}^{i}F(Y) \cong \boldsymbol{R}^{i}F(Z)$ for all $i \ge 0$.

(3) Put $Z'^{i} = \mathrm{Cok}(X^{i-1} \to X^{i})$ for $-n \le i < 0$. Then, since $\boldsymbol{R}^{0}F$ is left exact, $\boldsymbol{R}^{0}F(Z'^{i})$ embeds in $\boldsymbol{R}^{0}F(X^{i+1}) = 0$ for $-n \le i < 0$. Thus $\boldsymbol{R}^{i}F(X^{-n}) \cong \cdots \cong \boldsymbol{R}^{0}F(Z'^{-n+i}) = 0$ for $0 \le i < n$.

(4) We have $\boldsymbol{R}^{n+1}F(X) = \boldsymbol{R}^{n-1}F(Z) = 0$.

Claim 6: (1) For any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ there exists a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet \in \mathrm{Ob}(K(\mathscr{P}))$.

(2) $QG : K(\mathscr{P}) \to D(\mathscr{B})$ vanishes on the acyclic complexes.

Proof. (1) By Claim 5 and Lemma 16.6.

(2) By Claim 5 and Lemma 16.7.

Claim 7: $\boldsymbol{L}G$ exists.

Proof. By Claim 6 and Proposition 14.6.

Claim 8: There exists an isomorphism of ∂-functors $\eta : \boldsymbol{R}F \xrightarrow{\sim} \boldsymbol{L}G \circ T^{-n}$.

Proof. Let $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$. Let $\mu : X^\bullet \to C^{\bullet\bullet}$ be a right Cartan-Eilenberg resolution of $X^\bullet$ and put $I^{\bullet\bullet} = \sigma^{\mathrm{II}}_{\leq n}(C^{\bullet\bullet})$. Then by (1) of Claim 1 we have an exact sequence in $C(\mathscr{A})$

$$0 \to X^\bullet \xrightarrow{\mu} I^{\bullet,0} \to \cdots \to I^{\bullet,n-1} \to I^{\bullet,n} \to 0$$

with $I^{\bullet,0}, \cdots, I^{\bullet,n-1} \in \mathrm{Ob}(K(\mathscr{I}))$. Thus by (3) of Claim 1 $I^{\bullet,n} \in \mathrm{Ob}(K(\mathscr{I}))$, so that $t(I^{\bullet\bullet}) \in \mathrm{Ob}(K(\mathscr{I}))$. Also, by Proposition 15.12 $t(\mu) : X^\bullet \to t(I^{\bullet\bullet})$ is a quasi-isomorphism. Thus

$$\begin{aligned} \boldsymbol{R}F(Q(X^\bullet)) &\cong \boldsymbol{R}F(Q(t(I^{\bullet\bullet}))) \\ &\cong Q(F(t(I^{\bullet\bullet}))) \\ &\cong Q(t(F(I^{\bullet\bullet}))). \end{aligned}$$

Applying F to a right resolution

$$0 \to X^\bullet \xrightarrow{\mu} I^{\bullet,0} \to \cdots \to I^{\bullet,n-1} \to I^{\bullet,n} \to 0,$$

we get a complex in $C(\mathscr{A})$

$$\cdots \to 0 \to F(I^{\bullet,0}) \to \cdots \to F(I^{\bullet,n-1}) \to F(I^{\bullet,n}) \xrightarrow{\varepsilon} G(X^\bullet) \to 0 \to \cdots,$$

namely, we get a morphism $\varepsilon : F(I^{\bullet\bullet}) \to T_2^{-n}(G(X^\bullet))$ in $C^2(\mathscr{A})$. Thus we get a morphism

$$t(\varepsilon) : t(F(I^{\bullet\bullet})) \to t(T_2^{-n}(G(X^\bullet))) = G(T^{-n}(X^\bullet))$$

in $C(\mathscr{A})$. Consequently, we get a morphism in $D(\mathscr{B})$

$$\zeta_X : \boldsymbol{R}F(Q(X^\bullet)) \to Q(G(T^{-n}(X^\bullet))).$$

It follows by Lemmas 15.5, 15.7, 15.8, 15.9 and 15.10 that ζ_X is natural in $X^\bullet$. Thus we have a homomorphism of ∂-functors $\zeta \in \operatorname{Hom}(\boldsymbol{R}F \circ Q, Q \circ G \circ T^{-n})$. Note that $\boldsymbol{L}G \circ T^{-n}$ is a left derived functor of $G \circ T^{-n}$. Let $\xi : \boldsymbol{L}G \circ T^{-n} \circ Q \to Q \circ G \circ T^{-n}$ be the canonical homomorphism. Since we have an isomorphism

$$\operatorname{Hom}(\boldsymbol{R}F, \boldsymbol{L}G \circ T^{-n}) \xrightarrow{\sim} \operatorname{Hom}(\boldsymbol{R}F \circ Q, Q \circ G \circ T^{-n}),\ \eta \mapsto \xi \circ \eta_Q,$$

there exists a unique $\eta \in \operatorname{Hom}(\boldsymbol{R}F, \boldsymbol{L}G \circ T^{-n})$ with $\zeta = \xi \circ \eta_Q$. Consider now the case $X^\bullet \in \operatorname{Ob}(K(\mathscr{P}))$. Then $\varepsilon : F(I^{\bullet\bullet}) \to T_2^{-n}(G(X^\bullet))$ is a left resolution and by Proposition 15.17 $t(\varepsilon)$ is a quasi-isomorphism. Thus ζ_X is an isomorphism. Then, since by Proposition 14.6 ξ_X is an isomorphism, so is η_{QX}. Since the canonical functor $Q : K(\mathscr{P}) \to D(\mathscr{A})$ is dense, it follows that η is an isomorphism.

Proposition 16.12 (Dual of Proposition 16.11). *Assume $\mathscr{A}$ has enough projectives and let $G : \mathscr{A} \to \mathscr{B}$ be a right exact functor. Assume G has homological dimension $\leq n$ on $\mathscr{A}$, i.e., $\boldsymbol{L}_i G$ vanishes on $\mathscr{A}$ for $i > n$ (Note that by Corollary 14.7 $\boldsymbol{L}^- G$ exists). Put $F = \boldsymbol{L}_n G : \mathscr{A} \to \mathscr{B}$. Let $\mathscr{I}$ be the collection of $X \in \operatorname{Ob}(\mathscr{A})$ with $\boldsymbol{L}_i G(X) = 0$ for $i \quad n$ and assume that for any $X \in \operatorname{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$. Then both $\boldsymbol{L}G$ and $\boldsymbol{R}F$ exist and there exists an isomorphism of ∂-functors $\eta : \boldsymbol{R}F \circ T^n \xrightarrow{\sim} \boldsymbol{L}G$.*

§17. Derived functors of bi-∂-functors

Throughout this section, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ are abelian categories and $K^*(\mathscr{A})$, $K^\dagger(\mathscr{B})$ are localizing subcategories of $K(\mathscr{A})$ and $K(\mathscr{B})$, respectively. Unless stated otherwise, bifunctors are contravariant in the first variable and covariant in the second variable.

Definition 17.1. A bi-∂-functor $F = (F, \alpha, \beta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ is a bifunctor $F : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ together with isomorphisms of bifunctors

$$\alpha : F \circ (\mathbf{1} \times T) \xrightarrow{\sim} T \circ F, \quad \beta : F \circ (T^{-1} \times \mathbf{1}) \xrightarrow{\sim} T \circ F$$

such that

$$F(X^\bullet, -) = (F(X^\bullet, -), \alpha_{(X, -)}) : K^\dagger(\mathscr{B}) \to K(\mathscr{C}),$$
$$F(-, M^\bullet) = (F(-, M^\bullet), \beta_{(-, M)}) : K^*(\mathscr{A}) \to K(\mathscr{C})$$

are ∂-functors for all $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$.

The same definiton is also made for bifunctors of the form

$$K^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C}),\ D^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to D(\mathscr{C}) \text{ and } D^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C}).$$

Remark 17.1. For a bi-∂-functor $F = (F, \alpha, \beta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$, the following are equivalent (cf. Proposition 7.8(5)).

(1) $T\alpha \circ \beta_{(\mathbf{1} \times T)} + T\beta \circ \alpha_{(T^{-1} \times \mathbf{1})} = 0$. This is the case if $F = \mathrm{Hom}^\bullet$ or $\otimes$ (see Lemmas 18.2 and 19.2, respectively).

(2) For any $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$,

$$\alpha_{(-, M)} : F(-, TM^\bullet) \xrightarrow{\sim} T \circ F(-, M^\bullet)$$

is an isomorphism of ∂-functors.

(3) For any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$,

$$\beta_{(X, -)} : F(T^{-1} X^\bullet, -) \xrightarrow{\sim} T \circ F(X^\bullet, -)$$

is an isomorphism of ∂-functors.

Remark 17.2. Let $F = (F, \alpha, \beta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ be a bi-∂-functor. Then by Proposition 7.9(1)

$$\alpha_{(X,-)} : F(X^\bullet, -) \circ T \xrightarrow{\sim} T \circ F(X^\bullet, -), \quad \beta_{(-,M)} : F(-, M^\bullet) \circ T^{-1} \xrightarrow{\sim} T \circ F(-, M^\bullet)$$

are isomorphisms of ∂-functors for all $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$.

Remark 17.3. Let $F = (F, \alpha, \beta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ be a bi-∂-functor. Then the following hold.

(1) If $H = (H, \theta) : K(\mathscr{C}) \to K(\mathscr{D})$ is a ∂-functor, then

$$HF = (HF, \theta_F \circ H\alpha, T^{-1}\theta_{TF} \circ H\beta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{D})$$

is a bi-∂-functor.

(2) If $K^\#(\mathscr{D})$ is a localizing subcategory of $K(\mathscr{D})$ and $H = (H, \theta) : K^\#(\mathscr{D}) \to K^\dagger(\mathscr{B})$ is a ∂-functor, then

$$F \circ (\mathbf{1} \times H) = (F \circ (\mathbf{1} \times H), \alpha_{(\mathbf{1}\times H)} \circ F(1 \times \theta), \beta_{(\mathbf{1}\times H)}) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\#(\mathscr{D}) \to K(\mathscr{C})$$

is a bi-∂-functor.

(3) If $K^\#(\mathscr{D})$ is a localizing subcategory of $K(\mathscr{D})$ and $H = (H, \theta) : K^\#(\mathscr{D}) \to K^*(\mathscr{A})$ is a ∂-functor, then

$$F \circ (H \times \mathbf{1}) = (F \circ (H \times \mathbf{1}), \alpha_{(H\times\mathbf{1})}, \beta_{(H\times\mathbf{1})} \circ F(\theta \times 1)) : K^\#(\mathscr{D})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$$

is a bi-∂-functor.

Definition 17.2. Let $(F, \alpha, \beta), (G, \gamma, \delta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ be bi-∂-functors. A homomorphism of bi-∂-functors $\zeta : (F, \alpha, \beta) \to (G, \gamma, \delta)$ is a homomorphism of bifunctors $\zeta : F \to G$ such that $T\zeta \circ \alpha = \gamma \circ \zeta_{(\mathbf{1}\times T)}$ and $T\zeta \circ \beta = \delta \circ \zeta_{(T^{-1}\times\mathbf{1})}$.

Remark 17.4. Let (F, α, β) and $(G, \gamma, \delta) : K^*(\mathscr{A})^{\mathrm{op}} \times K^\dagger(\mathscr{B}) \to K(\mathscr{C})$ be bi-∂-functors and $\zeta : (F, \alpha, \beta) \to (G, \gamma, \delta)$ a homomorphism of bi-∂-functors. Then the following hold.

(1) For any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, $\zeta_{(X,-)} : F(X^\bullet, -) \to G(X^\bullet, -)$ is a homomorphism of ∂-functors.

(2) For any $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$, $\zeta_{(-,M)} : F(-, M^\bullet) \to G(-, M^\bullet)$ is a homomorphism of ∂-functors.

(3) If $H : K(\mathscr{C}) \to K(\mathscr{D})$ is a ∂-functor, then $H\zeta : HF \to HG$ is a homomorphism of bi-∂-functors.

(4) If $K^\#(\mathscr{D})$ is a localizing subcategory of $K(\mathscr{D})$ and $H : K^\#(\mathscr{D}) \to K^\dagger(\mathscr{B})$ is a ∂-functor, then $\zeta_{(\mathbf{1}\times H)} : F \circ (\mathbf{1} \times H) \to G \circ (\mathbf{1} \times H)$ is a homomorphism of bi-∂-functors.

(5) If $K^{\#}(\mathscr{D})$ is a localizing subcategory of $K(\mathscr{D})$ and $H : K^{\#}(\mathscr{D}) \to K^{*}(\mathscr{A})$ is a ∂-functor, then $\zeta_{(H\times\mathbf{1})} : F \circ (H\times\mathbf{1}) \to G \circ (H\times\mathbf{1})$ is a homomorphism of bi-∂-functors.

Definition 17.3. Let $F : K^{*}(\mathscr{A})^{\mathrm{op}} \times K^{\dagger}(\mathscr{B}) \to K(\mathscr{C})$ be a bi-∂-functor. A right derived functor of F is an initial object of the following category: an object is a pair (ζ, G) of a bi-∂-functor $G : D^{*}(\mathscr{A})^{\mathrm{op}} \times D^{\dagger}(\mathscr{B}) \to D(\mathscr{C})$ and a homomorphism of bi-∂-functors $\zeta : Q \circ F \to G \circ (Q \times Q)$; and a morphism $\eta : (\zeta_1, G_1) \to (\zeta_2, G_2)$ is a homomorphism of bi-∂-functors $\eta : G_1 \to G_2$ such that $\zeta_2 = \eta_{(Q\times Q)} \circ \zeta_1$.

Throughout the rest of this section,

$$F = (F, \alpha, \beta) : K^{*}(\mathscr{A})^{\mathrm{op}} \times K^{\dagger}(\mathscr{B}) \to K(\mathscr{C})$$

stands for a bi-∂-functor.

Lemma 17.1. *Assume the following conditions*:

(a) $F(X^{\bullet}, -)$ *has a right derived functor* $((\xi_{\mathrm{II}})_{(X,-)}, \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -))$ *for all* $X^{\bullet} \in \mathrm{Ob}(K^{*}(\mathscr{A}))$; *and*

(b) $K^{\dagger}(\mathscr{B})$ *has a full triangulated subcategory* $\mathscr{M}$ *such that for any* $M^{\bullet} \in \mathrm{Ob}(K^{\dagger}(\mathscr{B}))$ *there exists a quasi-isomorphism* $M^{\bullet} \to I^{\bullet}$ *with* $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$ *and* $(\xi_{\mathrm{II}})_{(-,I)}$ *is an isomorphism for all* $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$.

Then the following hold.

(1) $\boldsymbol{R}_{\mathrm{II}}F : K^{*}(\mathscr{A})^{\mathrm{op}} \times D^{\dagger}(\mathscr{B}) \to D(\mathscr{C})$ *is a bi-∂-functor*.

(2) $\xi_{\mathrm{II}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}F \circ (\mathbf{1} \times Q)$ *is a homomorphism of bi-∂-functors.*

Proof. We divide the proof into several steps.

Claim 1: $\boldsymbol{R}_{\mathrm{II}}F : K^{*}(\mathscr{A})^{\mathrm{op}} \times D^{\dagger}(\mathscr{B}) \to D(\mathscr{C})$ is a bifunctor and $\xi_{\mathrm{II}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}F \circ (\mathbf{1} \times Q)$ is a homomorphism of bifunctors.

Proof. It follows by Corollary 13.3(2) that for any $u \in K^{*}(\mathscr{A})(X^{\bullet}, Y^{\bullet})$ there exists a unique homomorphism of ∂-functors

$$\boldsymbol{R}_{\mathrm{II}}F(u, -) : \boldsymbol{R}_{\mathrm{II}}F(Y^{\bullet}, -) \to \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -)$$

such that $(\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(u, -)) = \boldsymbol{R}_{\mathrm{II}}F(u, -)_Q \circ (\xi_{\mathrm{II}})_{(Y,-)}$. Then for any $X^{\bullet} \in \mathrm{Ob}(K^{*}(\mathscr{A}))$, since $(\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(\mathrm{id}_X, -)) = (\xi_{\mathrm{II}})_{(X,-)}$, we have $\boldsymbol{R}_{\mathrm{II}}F(\mathrm{id}_X, -) = \mathrm{id}$. Also, for any two consecutive morphisms $u \in K^{*}(\mathscr{A})(X^{\bullet}, Y^{\bullet})$ and $v \in K^{*}(\mathscr{A})(Y^{\bullet}, Z^{\bullet})$, since

$$
\begin{aligned}
(\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(vu,-)) &= (\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(u,-) \circ F(v,-)) \\
&= (\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(u,-)) \circ Q(F(v,-)) \\
&= \boldsymbol{R}_{\mathrm{II}}F(u,-)_Q \circ (\xi_{\mathrm{II}})_{(Y,-)} \circ Q(F(v,-)) \\
&= \boldsymbol{R}_{\mathrm{II}}F(u,-)_Q \circ \boldsymbol{R}_{\mathrm{II}}F(v,-)_Q \circ (\xi_{\mathrm{II}})_{(Z,-)} \\
&= (\boldsymbol{R}_{\mathrm{II}}F(u,-) \circ \boldsymbol{R}_{\mathrm{II}}F(v,-))_Q \circ (\xi_{\mathrm{II}})_{(Z,-)},
\end{aligned}
$$

we have $\boldsymbol{R}_{\mathrm{II}}F(u,-) \circ \boldsymbol{R}_{\mathrm{II}}F(v,-) = \boldsymbol{R}_{\mathrm{II}}F(vu,-)$. Thus $\boldsymbol{R}_{\mathrm{II}}F$ is a bifunctor. We have seen that

$$(\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(u,-)) = \boldsymbol{R}_{\mathrm{II}}F(u,-)_Q \circ (\xi_{\mathrm{II}})_{(Y,-)}$$

for all $u \in K^*(\mathscr{A})(X^\bullet, Y^\bullet)$. Thus ξ_{II} is a homomorphism of bifunctors.

Claim 2: There exists a unique isomorphism of bifunctors $\phi : \boldsymbol{R}_{\mathrm{II}}F \circ (\mathbf{1} \times T) \tilde{\to} T \circ \boldsymbol{R}_{\mathrm{II}}F$ such that $T(\xi_{\mathrm{II}}) \circ Q(\alpha) = \phi_{(\mathbf{1} \times Q)} \circ (\xi_{\mathrm{II}})_{(\mathbf{1} \times T)}$. Furthermore, for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$,

$$\boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-) = (\boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-), \phi_{(X,-)}) : D^\dagger(\mathscr{B}) \to D(\mathscr{C})$$

is a ∂-functor.

Proof. Let $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$. Since we have an isomorphism of ∂-functors

$$Q(\alpha_{(X,-)}) : Q \circ F(X^\bullet,-) \circ T \tilde{\to} Q \circ T \circ F(X^\bullet,-) = T \circ Q \circ F(X^\bullet,-),$$

there exists a unique homomorphism of ∂-functors

$$\phi_{(X,-)} : \boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-) \circ T \to T \circ \boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-)$$

such that $T((\xi_{\mathrm{II}})_{(X,-)}) \circ Q(\alpha_{(X,-)}) = (\phi_{(X,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T$, so that by Corollary 13.3(1)

$$\boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-) = (\boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-), \phi_{(X,-)}) : D^\dagger(\mathscr{B}) \to D(\mathscr{C})$$

is a ∂-functor. Similarly, there exists a unique homomorphism of ∂-functors

$$\varphi_{(X,-)} : T \circ \boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-) \to \boldsymbol{R}_{\mathrm{II}}F(X^\bullet,-) \circ T$$

such that $((\xi_{\mathrm{II}})_{(X,-)})_T \circ Q(\alpha_{(X,-)})^{-1} = (\varphi_{(X,-)})_Q \circ T((\xi_{\mathrm{II}})_{(X,-)})$. Then

$$(\varphi_{(X,-)} \circ \phi_{(X,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T = \varphi_{(X,-)Q} \circ (\phi_{(X,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T$$

$$= \varphi_{(X,-)Q} \circ T((\xi_{\mathrm{II}})_{(X,-)}) \circ Q(\alpha_{(X,-)})$$
$$= ((\xi_{\mathrm{II}})_{(X,-)})_T \circ Q(\alpha_{(X,-)})^{-1} \circ Q(\alpha_{(X,-)})$$
$$= ((\xi_{\mathrm{II}})_{(X,-)})_T$$

and by Proposition 13.2 $\varphi_{(X,-)} \circ \phi_{(X,-)} = \mathrm{id}$. Similarly, $\phi_{(X,-)} \circ \varphi_{(X,-)} = \mathrm{id}$. Thus $\phi_{(X,-)}$ is an isomorphism. Next, let $u \in K^*(\mathscr{A})(X^\bullet, Y^\bullet)$. Then

$$\begin{aligned}(T(\boldsymbol{R}_{\mathrm{II}}F(u,-)) \circ \phi_{(Y,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T &= T(\boldsymbol{R}_{\mathrm{II}}F(u,-))_Q \circ (\phi_{(Y,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T\\ &= T(\boldsymbol{R}_{\mathrm{II}}F(u,-))_Q \circ T((\xi_{\mathrm{II}})_{(Y,-)}) \circ Q(\alpha_{(Y,-)})\\ &= T((\xi_{\mathrm{II}})_{(X,-)}) \circ T(QF(u,-)) \circ Q(\alpha_{(Y,-)})\\ &= T((\xi_{\mathrm{II}})_{(X,-)}) \circ Q(\alpha_{(X,-)}) \circ Q(F(u,-))_T\\ &= (\phi_{(X,-)})_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T \circ Q(F(u,-))_T\\ &= (\phi_{(X,-)})_Q \circ \boldsymbol{R}_{\mathrm{II}}F(u,-)_{QT} \circ ((\xi_{\mathrm{II}})_{(X,-)})_T\\ &= (\phi_{(X,-)} \circ \boldsymbol{R}_{\mathrm{II}}F(u,-)_T)_Q \circ ((\xi_{\mathrm{II}})_{(X,-)})_T,\end{aligned}$$

so that $T(\boldsymbol{R}_{\mathrm{II}}F(u,-)) \circ \phi_{(Y,-)} = \phi_{(X,-)} \circ \boldsymbol{R}_{\mathrm{II}}F(u,-)_T$. It follows that ϕ is an isomorphism of bifunctors and $T(\xi_{\mathrm{II}}) \circ Q(\alpha) = \phi_{(\mathbf{1}\times Q)} \circ (\xi_{\mathrm{II}})_{(\mathbf{1}\times T)}$.

Claim 3: There exists a unique isomorphism of bifunctors $\psi : \boldsymbol{R}_{\mathrm{II}}F \circ (T^{-1} \times \mathbf{1}) \xrightarrow{\sim} T \circ \boldsymbol{R}_{\mathrm{II}}F$ such that $T(\xi_{\mathrm{II}}) \circ Q(\beta) = \psi_{(\mathbf{1}\times Q)} \circ (\xi_{\mathrm{II}})_{(T^{-1}\times\mathbf{1})}$. Furthermore, for any $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathscr{B}))$,

$$\boldsymbol{R}_{\mathrm{II}}F(-, M^\bullet) = (\boldsymbol{R}_{\mathrm{II}}F(-, M^\bullet), \psi_{(-,M)}) : K^*(\mathscr{A}) \to D(\mathscr{C})$$

is a ∂-functor.

Proof. Let $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$. Take a quasi-isomorphism $s : M^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(\mathscr{M})$ and define an isomorphism of functors

$$\psi_{(-,QM)} : \boldsymbol{R}_{\mathrm{II}}F(-, QM^\bullet) \circ T^{-1} \xrightarrow{\sim} T \circ \boldsymbol{R}_{\mathrm{II}}F(-, QM^\bullet)$$

as a composite

$$\psi_{(-,QM)} = T(\boldsymbol{R}_{\mathrm{II}}F(-, Qs)^{-1} \circ (\xi_{\mathrm{II}})_{(-,I)}) \circ Q(\beta_{(-,I)}) \circ ((\xi_{\mathrm{II}})_{(-,I)})^{-1} \circ \boldsymbol{R}_{\mathrm{II}}F(-, Qs))_{T^{-1}}.$$

Then, as in the proof of Proposition 13.6, Lemma 13.5 enables us to see that $\psi_{(-,QM)}$ does not depend on the choice of s and

$$T(\boldsymbol{R}_{\mathrm{II}}F(-, Qu)) \circ \psi_{(-, QM)} = \psi_{(-, QN)} \circ \boldsymbol{R}_{\mathrm{II}}F(-, Qu)_{T^{-1}}$$

for all $u \in K^{\dagger}(\mathscr{B})(M^{\bullet}, N^{\bullet})$. Thus ψ is an isomorphism of bifunctors. Next, since

$$\psi_{(-, QI)} = T((\xi_{\mathrm{II}})_{(-, I)}) \circ Q(\beta_{(-, I)}) \circ ((\xi_{\mathrm{II}})_{(-, I)}{}^{-1})_{T^{-1}}$$

for all $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$, it follows that $T(\xi_{\mathrm{II}}) \circ Q(\beta) = \psi_{(\mathbf{1} \times Q)} \circ (\xi_{\mathrm{II}})_{(T^{-1} \times \mathbf{1})}$ and ψ is unique. Finally, since

$$Q \circ F(-, I^{\bullet}) = (Q \circ F(-, I^{\bullet}), Q(\beta_{(-, I)})) : K^{*}(\mathscr{A}) \to D(\mathscr{C})$$

is a ∂-functor, it follows by Proposition 7.8(3) that

$$\boldsymbol{R}_{\mathrm{II}}F(-, Q M^{\bullet}) = (\boldsymbol{R}_{\mathrm{II}}F(-, Q M^{\bullet}), \psi_{(-, QM)}) : K^{*}(\mathscr{A}) \to D(\mathscr{C})$$

is a ∂-functor.

Remark 17.5. If $T\alpha \circ \beta_{(\mathbf{1} \times T)} + T\beta \circ \alpha_{(T^{-1} \times \mathbf{1})} = 0$, then $T\phi \circ \psi_{(\mathbf{1} \times T)} + T\psi \circ \phi_{(T^{-1} \times \mathbf{1})} = 0$.

By symmetry, the following holds.

Lemma 17.2. *Assume the following conditions*:

(a) $F(-, M^{\bullet})$ *has a right derived functor* $((\xi_{\mathrm{I}})_{(-, M)}, \boldsymbol{R}_{\mathrm{I}}F(-, M^{\bullet}))$ *for all* $M^{\bullet} \in \mathrm{Ob}(K^{\dagger}(\mathscr{B}))$; *and*

(b) $K^{*}(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^{\bullet} \in \mathrm{Ob}(K^{\dagger}(\mathscr{B}))$ *there exists a quasi-isomorphism* $P^{\bullet} \to X^{\bullet}$ *with* $P^{\bullet} \in \mathrm{Ob}(\mathscr{L})$ *and* $(\xi_{\mathrm{I}})_{(P, -)}$ *is an isomorphism for all* $P^{\bullet} \in \mathrm{Ob}(\mathscr{L})$.

Then the following hold.

(1) $\boldsymbol{R}_{\mathrm{I}}F : D^{*}(\mathscr{A})^{\mathrm{op}} \times K^{\dagger}(\mathscr{B}) \to D(\mathscr{C})$ *is a bi-∂-functor.*

(2) $\xi_{\mathrm{I}} : Q \circ F \to \boldsymbol{R}_{\mathrm{I}}F \circ (Q \times \mathbf{1})$ *is a homomorphism of bi-∂-functors*.

Proposition 17.3. *Assume the following conditions*:

(a) $F(X^{\bullet}, -)$ *has a right derived functor* $((\xi_{\mathrm{II}})_{(X, -)}, \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -))$ *for all* $X^{\bullet} \in \mathrm{Ob}(K^{*}(\mathscr{A}))$;

(b) $K^{\dagger}(\mathscr{B})$ *has a full triangulated subcategory* $\mathscr{M}$ *such that for any* $M^{\bullet} \in \mathrm{Ob}(K^{\dagger}(\mathscr{B}))$ *there exists a quasi-isomorphism* $M^{\bullet} \to I^{\bullet}$ *with* $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$ *and* $(\xi_{\mathrm{II}})_{(-, I)}$ *is an isomorphism for all* $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$;

(c) $\boldsymbol{R}_{\mathrm{II}}F(-, M^{\bullet})$ *has a right derived functor* $((\xi_{\mathrm{I}})_{(-, M)}, \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-, M^{\bullet}))$ *for all* $M^{\bullet} \in \mathrm{Ob}(D^{\dagger}(\mathscr{B}))$; *and*

(d) $K^{*}(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^{\bullet} \in \mathrm{Ob}(K^{\dagger}(\mathscr{B}))$ *there* *exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$ *and* $(\xi_{\mathrm{I}})_{(P,-)}$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$.

Then the following hold.

(1) $\boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F : D^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C})$ *is a bi-∂-functor.*

(2) $\xi = (\xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F \circ (Q \times Q)$ *is a homomorphism of bi–∂-functors.*

(3) $(\xi, \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F)$ *is a right derived functor of* F.

Proof. We use the same notation as in the proof of Lemma 17.1. By Lemma 17.1 $\boldsymbol{R}_{\mathrm{II}}F = (\boldsymbol{R}_{\mathrm{II}}F, \phi, \psi)$ is a bi-∂-functor and

$$\xi_{\mathrm{II}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}F \circ (\mathbf{1} \times Q)$$

is a homomorphism of bi-∂-functors. Also, applying Lemma 17.2 to $\boldsymbol{R}_{\mathrm{II}}F$, we conclude that $\boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F$ is a bi-∂-functor and

$$\xi_{\mathrm{I}} : \boldsymbol{R}_{\mathrm{II}}F \to \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F \circ (Q \times \mathbf{1})$$

is a homomorphism of bi–∂-functors. Then, since

$$(\xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} : \boldsymbol{R}_{\mathrm{II}}F \circ (\mathbf{1} \times Q) \to \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F \circ (Q \times Q)$$

is also a homomorphism of bi–∂-functors, so is the composite

$$\xi = (\xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F \circ (Q \times Q).$$

Next, let (ζ, G) be a pair of a bi-∂-functor $G = (G, \gamma, \delta)$: $D^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C})$ and a homomorphism of bi-∂-functors $\zeta : Q \circ F \to G \circ (Q \times Q)$. We claim that there exists a unique homomorphism of bi-∂-functors $\eta : (\xi, \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F) \to (\zeta, G)$ such that $\zeta = \eta_{(Q\times Q)} \circ \xi$. We divide the proof into several steps.

Claim 1: There exists a unique homomorphism of bifunctors $\kappa : \boldsymbol{R}_{\mathrm{II}}F \to G \circ (Q \times \mathbf{1})$ such that $\zeta = \kappa_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}}$. Furthermore, for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$,

$$\kappa_{(X,-)} : \boldsymbol{R}_{\mathrm{II}}F(X^\bullet, -) \to G(QX^\bullet, -)$$

is a homomorphism of ∂-functors.

Proof. For any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, since we have a homomorphism of ∂-functors

$$\zeta_{(X,-)} : Q \circ F(X^{\bullet}, -) \to G(QX^{\bullet}, -) \circ Q,$$

there exists a unique homomorphism of ∂-functors

$$\kappa_{(X,-)} : \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -) \to G(QX^{\bullet}, -)$$

such that $\zeta_{(X,-)} = (\kappa_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(X,-)}$. Then for any $u \in K^*(\mathscr{A})(X^{\bullet}, Y^{\bullet})$, since

$$\begin{aligned}
(\kappa_{(X,-)} \circ \boldsymbol{R}_{\mathrm{II}}F(u,-))_Q \circ (\xi_{\mathrm{II}})_{(Y,-)} &= (\kappa_{(X,-)})_Q \circ \boldsymbol{R}_{\mathrm{II}}F(u,-)_Q \circ (\xi_{\mathrm{II}})_{(Y,-)} \\
&= (\kappa_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(X,-)} \circ Q(F(u,-)) \\
&= \zeta_{(X,-)} \circ Q(F(u,-)) \\
&= G(Q(u),-)_Q \circ \zeta_{(Y,-)} \\
&= G(Q(u),-)_Q \circ (\kappa_{(Y,-)})_Q \circ (\xi_{\mathrm{II}})_{(Y,-)} \\
&= (G(Q(u),-) \circ \kappa_{(Y,-)})_Q \circ (\xi_{\mathrm{II}})_{(Y,-)},
\end{aligned}$$

by Proposition 13.2 $\kappa_{(X,-)} \circ \boldsymbol{R}_{\mathrm{II}}F(u,-) = G(Q(u),-) \circ \kappa_{(Y,-)}$. Thus κ is a homomorphism of bifunctors. It then follows that $\zeta = \kappa_{(\mathbf{1} \times Q)} \circ \xi_{\mathrm{II}}$. Also, since

$$\kappa_{(-,QI)} = \zeta_{(-,I)} \circ (\xi_{\mathrm{II}})_{(-,I)}^{-1}$$

for all $I^{\bullet} \in \mathrm{Ob}(\mathscr{M})$, it follows that κ is unique.

Claim 2: $\kappa : \boldsymbol{R}_{\mathrm{II}}F \to G \circ (Q \times \mathbf{1})$ is a homomorphism of bi-∂-functors.

Proof. Let $X^{\bullet} \in \mathrm{Ob}(K^*(\mathscr{A}))$. Since $\kappa_{(X,-)} : \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -) \to G(QX^{\bullet}, -)$ is a homomorphism of ∂-functors, we have $T(\kappa_{(X,-)}) \circ \phi_{(X,-)} = \gamma_{(QX,-)} \circ (\kappa_{(X,-)})_T$. Thus $T\kappa \circ \phi = \gamma_{(Q \times \mathbf{1})} \circ \kappa_{(\mathbf{1} \times T)}$. It remains to see that $T\kappa \circ \psi = \delta_{(Q \times \mathbf{1})} \circ \kappa_{(T^{-1} \times \mathbf{1})}$. We have isomorphisms of ∂-functors

$$\begin{aligned}
Q(\beta_{(X,-)}) &: Q \circ F(T^{-1}X^{\bullet}, -) \xrightarrow{\sim} T \circ Q \circ F(X^{\bullet}, -), \\
(\psi_{(X,-)})_Q &: \boldsymbol{R}_{\mathrm{II}}F(T^{-1}X^{\bullet}, -) \circ Q \xrightarrow{\sim} T \circ \boldsymbol{R}_{\mathrm{II}}F(X^{\bullet}, -) \circ Q, \\
(\delta_{(QX,-)})_Q &: G(T^{-1}QX^{\bullet}, -) \circ Q \xrightarrow{\sim} T \circ G(QX^{\bullet}, -) \circ Q.
\end{aligned}$$

Also, since ξ_{II} and ζ are homomorphisms of bi-∂-functors, we have

$$\begin{aligned}
T((\xi_{\mathrm{II}})_{(X,-)}) \circ Q(\beta_{(X,-)}) &= (\psi_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(T^{-1}X,-)}, \\
T(\zeta_{(X,-)}) \circ Q(\beta_{(X,-)}) &= (\delta_{(QX,-)})_Q \circ \zeta_{(T^{-1}X,-)}.
\end{aligned}$$

Thus

$$
\begin{aligned}
((T\kappa \circ \psi)_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(T^{-1}X,-)} &= T((\kappa_{(\mathbf{1}\times Q)})_{(X,-)}) \circ (\psi_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(T^{-1}X,-)} \\
&= T((\kappa_{(\mathbf{1}\times Q)})_{(X,-)}) \circ T((\xi_{\mathrm{II}})_{(X,-)}) \circ Q(\beta_{(X,-)}) \\
&= T((\kappa_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}})_{(X,-)}) \circ Q(\beta_{(X,-)}) \\
&= T(\zeta_{(X,-)}) \circ Q(\beta_{(X,-)}) \\
&= (\delta_{(QX,-)})_Q \circ \zeta_{(T^{-1}X,-)} \\
&= (\delta_{(QX,-)})_Q \circ (\kappa_{(\mathbf{1}\times Q)} \circ (\xi_{\mathrm{II}}))_{(T^{-1}X,-)} \\
&= (\delta_{(QX,-)})_Q \circ (\kappa_{(\mathbf{1}\times Q)})_{(T^{-1}X,-)} \circ (\xi_{\mathrm{II}})_{(T^{-1}X,-)} \\
&= ((\delta_{(Q\times\mathbf{1})} \circ \kappa_{(T^{-1}\times\mathbf{1})})_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(T^{-1}X,-)}.
\end{aligned}
$$

It follows by Proposition 13.2 that $(T\kappa \circ \psi)_{(X,-)} = (\delta_{(Q\times\mathbf{1})} \circ \kappa_{(T^{-1}\times\mathbf{1})})_{(X,-)}$.

Claim 3: There exists a unique homomorphism of bifunctors $\eta : \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F \to G$ such that $\kappa = \eta_{(Q\times\mathbf{1})} \circ \xi_{\mathrm{I}}$. Furthermore, for any $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathscr{B}))$,

$$\eta_{(-,M)} : \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-, M^\bullet) \to G(-, M^\bullet)$$

is a homomorphism of ∂-functors for all and.

Proof. For any $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathscr{B}))$, since by Claim 2

$$\kappa_{(-,M)} : \boldsymbol{R}_{\mathrm{II}}F(-, M^\bullet) \to G(-, M^\bullet) \circ Q$$

is a homomorphism of ∂-functors, there exists a unique homomorphism of ∂-functors

$$\eta_{(-,M)} : \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-, M^\bullet) \to G(-, M^\bullet)$$

such that $\kappa_{(-,M)} = (\eta_{(-,M)})_Q \circ (\xi_{\mathrm{I}})_{(-,M)}$. Then for any $u \in D^\dagger(\mathscr{B})(M^\bullet, N^\bullet)$ we have

$$
\begin{aligned}
(\eta_{(-,N)} \circ \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-,u))_Q \circ (\xi_{\mathrm{I}})_{(-,M)} &= (\eta_{(-,N)})_Q \circ \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-,u)_Q \circ (\xi_{\mathrm{I}})_{(-,M)} \\
&= (\eta_{(-,N)})_Q \circ (\xi_{\mathrm{I}})_{(-,N)} \circ \boldsymbol{R}_{\mathrm{II}}F(-,u) \\
&= \kappa_{(-,N)} \circ \boldsymbol{R}_{\mathrm{II}}F(-,u) \\
&= G(-,u)_Q \circ \kappa_{(-,M)} \\
&= G(-,u)_Q \circ (\eta_{(-,M)})_Q \circ (\xi_{\mathrm{I}})_{(-,M)} \\
&= (G(-,u) \circ \eta_{(-,M)})_Q \circ (\xi_{\mathrm{I}})_{(-,M)}.
\end{aligned}
$$

Thus by Proposition 13.2 $\eta_{(-,N)} \circ \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F(-, u) = G(-, u) \circ \eta_{(-,M)}$. Hence η is a homomorphism of bifunctors and satisfies $\kappa = \eta_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}}$. Also, since

$$\eta_{(QP,-)} = \kappa_{(P,-)} \circ (\xi_{\mathrm{I}})_{(P,-)}^{-1}$$

for all $P^\bullet \in \mathrm{Ob}(\mathscr{L})$, it follows that η is unique.

Claim 4: η is a homomorphism of bi-∂-functors and satisfies $\zeta = \eta_{(Q\times Q)} \circ \xi$.

Proof. Since $\kappa = \eta_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}}$ and both ξ_{I} and κ are homomorphisms of bi-∂-functors, it follows by the same argument as in the proof of Claim 2 that $\eta_{(Q\times \mathbf{1})}$ is also a homomorphism of bi-∂-functors. It then follows that η is a homomorphism of bi-∂-functors. Also, by Claims 1 and 3 we have $\zeta = \eta_{(Q\times Q)} \circ \xi$.

Claim 5: η is unique.

Proof. Let $\varphi : \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}F \to G$ be a homomorphism of bi-∂-functors with $\zeta = \varphi_{(Q\times Q)} \circ \xi$. Then

$$\begin{aligned}(\varphi_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}} &= \varphi_{(Q\times Q)} \circ (\xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}} \\ &= \varphi_{(Q\times Q)} \circ \xi \\ &= \zeta \\ &= \kappa_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}}.\end{aligned}$$

Thus, for any $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$, we have

$$((\varphi_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}})_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(X,-)} = (\kappa_{(X,-)})_Q \circ (\xi_{\mathrm{II}})_{(X,-)},$$

and by Proposition 13.2 $(\varphi_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}})_{(X,-)} = \kappa_{(X,-)}$. Hence $\varphi_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}} = \kappa$ and by Claim 3 $\varphi = \eta$.

Changing the order of taking right derived functors, we get the following.

Proposition 17.4. *Assume the following conditions*:

(a) $F(-, M^\bullet)$ *has a right derived functor* $((\xi_{\mathrm{I}})_{(-,M)}, \boldsymbol{R}_{\mathrm{I}}F(-, M^\bullet))$ *for all* $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$;

(b) $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$ *there exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$ *and* $(\xi_{\mathrm{I}})_{(P,-)}$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathscr{L})$;

(c) $\boldsymbol{R}_{\mathrm{I}}F(X^\bullet, -)$ *has a right derived functor* $((\xi_{\mathrm{II}})_{(X,-)}, \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F(X^\bullet, -))$ *for all* $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$; *and*

(d) $K^\dagger(\mathscr{B})$ *has a full triangulated subcategory* $\mathscr{M}$ *such that for any* $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$ *there* *exists a quasi-isomorphism* $M^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(\mathcal{M})$ *and* $(\xi_{\mathrm{II}})_{(-,\,I)}$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(\mathcal{M})$.

Then the following hold.

(1) $\boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F : D^*(\mathcal{A})^{\mathrm{op}} \times D^\dagger(\mathcal{B}) \to D(\mathcal{C})$ *is a bi-∂-functor*.

(2) $\xi = (\xi_{\mathrm{II}})_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}} : Q \circ F \to \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F \circ (Q \times Q)$ *is a homomorphism of bi–∂-functors*.

(3) $(\xi, \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}F)$ *is a right derived functor of* F.

Definition 17.4. Let $K^*(\mathcal{A})$, $K^\dagger(\mathcal{B})$ be localizing subcategories of $K(\mathcal{A})$ and $K(\mathcal{B})$, respectively, and $F : K^*(\mathcal{A})^{\mathrm{op}} \times K^\dagger(\mathcal{B}) \to K(\mathcal{C})$ a bi-∂-functor. A left derived functor of F is a terminal object of the following category: an object is a pair (G, ζ) of a bi-∂-functor $G : D^*(\mathcal{A})^{\mathrm{op}} \times D^\dagger(\mathcal{B}) \to D(\mathcal{C})$ and a homomorphism of bi-∂-functors $\zeta : G \circ (Q \times Q) \to Q \circ F$; and a morphism $\eta : (G_1, \zeta_1) \to (G_2, \zeta_2)$ is a homomorphism of bi-∂-functors $\eta : G_1 \to G_2$ such that $\zeta_1 = \zeta_2 \circ \eta_{(Q\times Q)}$.

Lemma 17.5 (dual of Lemma 17.1). *Assume the following conditions*:

(a) $F(X^\bullet, -)$ *has a left derived functor* $(\boldsymbol{L}_{\mathrm{II}}F(X^\bullet, -), (\xi_{\mathrm{II}})_{(X,\,-)})$ *for all* $X^\bullet \in \mathrm{Ob}(K^*(\mathcal{A}))$; *and*

(b) $K^\dagger(\mathcal{B})$ *has a full triangulated subcategory* $\mathcal{M}$ *such that for any* $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathcal{B}))$ *there exists a quasi-isomorphism* $P^\bullet \to M^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathcal{M})$ *and* $(\xi_{\mathrm{II}})_{(-,\,P)}$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathcal{M})$.

Then the following hold.

(1) $\boldsymbol{L}_{\mathrm{II}}F : K^*(\mathcal{A})^{\mathrm{op}} \times D^\dagger(\mathcal{B}) \to D(\mathcal{C})$ *is a bi-∂-functor*.

(2) $\xi_{\mathrm{II}} : \boldsymbol{L}_{\mathrm{II}}F \circ (Q \times \mathbf{1}) \to Q \circ F$ *is a homomorphism of bi–∂-functors*.

Lemma 17.6 (dual of Lemma 17.2). *Assume the following conditions*:

(a) $F(-, M^\bullet)$ *has a left derived functor* $(\boldsymbol{L}_{\mathrm{I}}F(-, M^\bullet), (\xi_{\mathrm{I}})_{(-,\,M)})$ *for all* $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathcal{B}))$; *and*

(b) $K^*(\mathcal{A})$ *has a full triangulated subcategory* $\mathcal{L}$ *such that for any* $X^\bullet \in \mathrm{Ob}(K^\dagger(\mathcal{B}))$ *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(\mathcal{L})$ *and* $(\xi_{\mathrm{I}})_{(I,\,-)}$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(\mathcal{L})$.

Then the following hold.

(1) $\boldsymbol{L}_{\mathrm{I}}F : D^*(\mathcal{A})^{\mathrm{op}} \times K^\dagger(\mathcal{B}) \to D(\mathcal{C})$ *is a bi-∂-functor*.

(2) $\xi_{\mathrm{I}} : \boldsymbol{L}_{\mathrm{I}}F \circ (\mathbf{1} \times Q) \to Q \circ F$ *is a homomorphism of bi–∂-functors.*

Proposition 17.7 (dual of Proposition 17.3). *Assume the following conditions*:

(a) $F(X^\bullet, -)$ *has a left derived functor* $(\boldsymbol{L}_{\mathrm{II}}F(X^\bullet, -), (\xi_{\mathrm{II}})_{(X,\,-)})$ *for all* $X^\bullet \in \mathrm{Ob}(K^*(\mathcal{A}))$;

(b) $K^\dagger(\mathcal{B})$ *has a full triangulated subcategory* $\mathcal{M}$ *such that for any* $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathcal{B}))$ *there exists a quasi-isomorphism* $P^\bullet \to M^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathcal{M})$ *and* $(\xi_{\mathrm{II}})_{(-,\,P)}$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathcal{M})$;

(c) $\boldsymbol{L}_{\mathrm{II}}F(-, M^\bullet)$ *has a left derived functor* $((\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}F(-, M^\bullet), (\xi_{\mathrm{I}})_{(-,\,M)})$ *for all* $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathcal{B}))$; *and*

(d) $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$ *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(\mathscr{L})$ *and* $(\xi_{\mathrm{I}})_{(I,-)}$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(\mathscr{L})$.

Then the following hold.

(1) $\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}F : D^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C})$ *is a bi-∂-functor.*

(2) $\xi = (\xi_{\mathrm{II}})_{(Q\times \mathbf{1})} \circ \xi_{\mathrm{I}} : \boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}F \circ (Q \times Q) \to Q \circ F$ *is a homomorphism of bi–∂-functors.*

(3) $(\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}F, \xi)$ *is a left derived functor of F.*

Proposition 17.8 (dual of Proposition 17.4). *Assume the following conditions*:

(a) $F(-, M^\bullet)$ *has a left derived functor* $(\boldsymbol{L}_{\mathrm{I}}F(-, M^\bullet), (\xi_{\mathrm{I}})_{(-, M)})$ *for all* $M^\bullet \in \mathrm{Ob}(D^\dagger(\mathscr{B}))$;

(b) $K^*(\mathscr{A})$ *has a full triangulated subcategory* $\mathscr{L}$ *such that for any* $X^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$ *there exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(\mathscr{L})$ *and* $(\xi_{\mathrm{I}})_{(I,-)}$ *is an isomorphism for all* $I^\bullet \in \mathrm{Ob}(\mathscr{L})$;

(c) $\boldsymbol{L}_{\mathrm{I}}F(X^\bullet, -)$ *has a left derived functor* $((\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}F(X^\bullet, -), (\xi_{\mathrm{II}})_{(X,-)})$ *for all* $X^\bullet \in \mathrm{Ob}(K^*(\mathscr{A}))$; *and*

(d) $K^\dagger(\mathscr{B})$ *has a full triangulated subcategory* $\mathscr{M}$ *such that for any* $M^\bullet \in \mathrm{Ob}(K^\dagger(\mathscr{B}))$ *there exists a quasi-isomorphism* $P^\bullet \to M^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(\mathscr{M})$ *and* $(\xi_{\mathrm{II}})_{(-,P)}$ *is an isomorphism for all* $P^\bullet \in \mathrm{Ob}(\mathscr{M})$.

Then the following hold.

(1) $\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}F : D^*(\mathscr{A})^{\mathrm{op}} \times D^\dagger(\mathscr{B}) \to D(\mathscr{C})$ *is a bi-∂-functor.*

(2) $\xi = (\xi_{\mathrm{I}})_{(\mathbf{1}\times Q)} \circ \xi_{\mathrm{II}} : \boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}F \circ (Q \times Q) \to Q \circ F$ *is a homomorphism of bi–∂-functors.*

(3) $(\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}F, \xi)$ *is a left derived functor of F.*

§18. The right derived functor of $\mathrm{Hom}^\bullet$

Throught this section, $\mathscr{A}$ is an abelian category, $\mathscr{U}$ is the épaisse subcategory of $K(\mathscr{A})$ consisting of acyclic complexes and $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$. We denote by $K(\mathscr{I})_{\mathrm{L}}$ (resp. $K(\mathscr{P})_{\mathrm{L}}$) the the full subcategory of $K(\mathscr{I})$ (resp. $K(\mathscr{P})$) consisting of $\mathscr{U}$-local (resp. $\mathscr{U}$-colocal) complexes $X^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$.

Definition 18.1. For $X^\bullet$ and $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, we define a double complex $C^{\bullet\bullet}$ in $\mathrm{Mod}\,\mathbb{Z}$ as follows:

$$C^{p,q} = \mathscr{A}(X^{-p}, Y^q),$$
$$d_1^{p,q} = (-1)^{p+q+1}\,\mathscr{A}(d_X^{-(p+1)}, Y^q),$$
$$d_2^{p,q} = \mathscr{A}(X^{-p}, d_Y^q)$$

for $p, q \in \mathbb{Z}$, and set $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) = t(C^{\bullet\bullet})$. Then we get a bifunctor

$$\mathrm{Hom}^\bullet : C(\mathscr{A})^{\mathrm{op}} \times C(\mathscr{A}) \to C(\mathrm{Mod}\,\mathbb{Z})$$

such that

$$\mathrm{Hom}^n(X^\bullet, Y^\bullet) = \prod_{p+q=n} \mathscr{A}(X^{-p}, Y^q)$$

for $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ and $n \in \mathbb{Z}$, and

$$d^n_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}(u) = (-1)^{n+1}\, u \circ d_X^{-(p+1)} + d_Y^q \circ u$$

for $n \in \mathbb{Z}$, $p, q \in \mathbb{Z}$ with $p + q = n$ and $u \in \mathscr{A}(X^{-p}, Y^q)$.

Lemma 18.1. *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following hold.*

(1) *We may identify* $\mathrm{Hom}^n(X^\bullet, Y^\bullet)$ *with* $\mathscr{A}^{\mathbb{Z}}(X^\bullet, T^n(Y^\bullet))$ *for all* $n \in \mathbb{Z}$. *Then*

$$d^n_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}(u) = (-1)^{n+1}\{T(u) \circ d_X - d_{T^n Y} \circ u\} \in \mathscr{A}^{\mathbb{Z}}(X^\bullet, T^{n+1}(Y^\bullet))$$

for $n \in \mathbb{Z}$ *and* $u \in \mathscr{A}^{\mathbb{Z}}(X^\bullet, T^n(Y^\bullet))$. *In particular,* $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) \in \mathrm{Ob}(C(\mathrm{Mod}\,A^{\mathrm{op}}))$, *where* $A = \mathrm{End}_{C(\mathscr{A})}(X^\bullet)$.

(2) *We may identify* $\mathrm{Hom}^n(X^\bullet, Y^\bullet)$ *with* $\mathscr{A}^{\mathbb{Z}}(T^{-n}(X^\bullet), Y^\bullet)$ *for all* $n \in \mathbb{Z}$. *Then*

$$d^n_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}(u) = u \circ d_{T^{-(n+1)}X} + T^{-1}(d_Y \circ u) \in \mathscr{A}^{\mathbb{Z}}(T^{-(n+1)}(X^\bullet), Y^\bullet)$$

for $n \in \mathbb{Z}$ *and* $u \in \mathscr{A}^{\mathbb{Z}}(T^{-n}(X^\bullet), Y^\bullet)$. *In particular,* $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) \in \mathrm{Ob}(C(\mathrm{Mod}\, B))$, *where* $B = \mathrm{End}_{C(\mathscr{A})}(Y^\bullet)$.

Proof. (1) The first assertion is immediate by definition. Let $A = \mathrm{End}_{C(\mathscr{A})}(X^\bullet)$. Then, since A is a subring of $\mathrm{End}_{\mathscr{A}^{\mathbb{Z}}}(X^\bullet)$, every $\mathscr{A}^{\mathbb{Z}}(X^\bullet, T^n(Y^\bullet))$ is a right A-module. Also, it follows by the first assertion that every $d^n_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}$ is an A-linear map.

(2) Dual of (1).

Definition 18.2. For any abelian category $\mathscr{A}$ we denote by $\rho : \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$ an automorphism of the identity functor $\mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ such that $\rho^n_X = (-1)^n \mathrm{id}_{X^n}$ for all $X \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ and $n \in \mathbb{Z}$.

Lemma 18.2. (1) *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *we have*

$$d_{\mathrm{Hom}^\bullet(X^\bullet, TY^\bullet)} = -T(d_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}), \quad d_{\mathrm{Hom}^\bullet(T^{-1}X^\bullet, Y^\bullet)} = T(d_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}).$$

(2) *There exist isomorphisms of bifunctors*

$$\alpha : \mathrm{Hom}^\bullet \circ (\mathbf{1} \times T) \xrightarrow{\sim} T \circ \mathrm{Hom}^\bullet, \quad \beta : \mathrm{Hom}^\bullet \circ (T^{-1} \times \mathbf{1}) \xrightarrow{\sim} T \circ \mathrm{Hom}^\bullet$$

such that for any $X^\bullet$ *and* $Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$

$$\alpha_{(X, Y)} = \mathrm{id}_{\mathrm{Hom}^\bullet(X^\bullet, TY^\bullet)}, \quad \beta_{(X, Y)} = T(\rho_{\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)}).$$

In particular, $T\alpha \circ \beta_{(\mathbf{1} \times T)} + T\beta \circ \alpha_{(T^{-1} \times \mathbf{1})} = 0$.

Proof. (1) Straightforward.

(2) Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. Since

$$\mathrm{Hom}^n(X^\bullet, TY^\bullet) = \mathrm{Hom}^{n+1}(X^\bullet, Y^\bullet) = \mathrm{Hom}^n(T^{-1}X^\bullet, Y^\bullet)$$

for all $n \in \mathbb{Z}$, we may consider that

$$\mathrm{Hom}^\bullet(X^\bullet, TY^\bullet) = T(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}^\bullet(T^{-1}X^\bullet, Y^\bullet)$$

in $(\mathrm{Mod}\, \mathbb{Z})^{\mathbb{Z}}$. Thus by the part (1) we have natural isomorphisms in $C(\mathrm{Mod}\, \mathbb{Z})$

$$\alpha_{(X,\,Y)} = \mathrm{id}_{\mathrm{Hom}^\bullet(X^\bullet,\,TY^\bullet)} : \mathrm{Hom}^\bullet(X^\bullet, TY^\bullet) \xrightarrow{\sim} T(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)),$$
$$\beta_{(X,\,Y)} = T(\rho_{\mathrm{Hom}^\bullet(X^\bullet,\,Y^\bullet)}) : \mathrm{Hom}^\bullet(T^{-1}X^\bullet, Y^\bullet) \xrightarrow{\sim} T(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)).$$

It is obvious that $T\alpha \circ \beta_{(\mathbf{1}\times T)} + T\beta \circ \alpha_{(T^{-1}\times\mathbf{1})} = 0$.

Lemma 18.3. *For any* $X^\bullet, Y^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *and* $n \in \mathbb{Z}$, *we have*

$$H^n(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) \cong K(\mathscr{A})(X^\bullet, T^n(Y^\bullet)).$$

Proof. Let us identify $\mathrm{Hom}^n(X^\bullet, -)$ with $\mathscr{A}^{\mathbb{Z}}(X^\bullet, -) \circ T^n$ for all $n \in \mathbb{Z}$. Then we have

$$\begin{aligned} Z^n(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) &= \{u \in \mathscr{A}^{\mathbb{Z}}(X^\bullet, T^n(Y^\bullet)) \mid T(u) \circ d_X = d_{T^nY} \circ u\} \\ &= C(\mathscr{A})(X^\bullet, T^n(Y^\bullet)). \end{aligned}$$

Also, for any $v \in \mathscr{A}^{\mathbb{Z}}(X^\bullet, T^{n-1}(Y^\bullet))$, since

$$\begin{aligned} d^{n-1}(v) &= (-1)^n\{T(v) \circ d_X - d_{T^{n-1}Y} \circ v\} \\ &= (-1)^n\{T(v) \circ d_X + T^{-1}(d_{T^nY}) \circ T^{-1}(T(v))\} \\ &= (-1)^n\{T(v) \circ d_X + T^{-1}(d_{T^nY} \circ T(v))\} \end{aligned}$$

with $T(v) \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, T^n(Y^\bullet))$, we have $d^{n-1}(v) \in \mathrm{Htp}(X^\bullet, T^n(Y^\bullet))$. Conversely, for any $u \in \mathrm{Htp}(X^\bullet, T^n(Y^\bullet))$, since there exists $h \in \mathscr{A}^{\mathbb{Z}}(TX^\bullet, T^n(Y^\bullet))$ such that

$$\begin{aligned} u &= h \circ d_X + T^{-1}(d_{T^nY} \circ h) \\ &= (-1)^n d^{n-1}(T^{-1}(h)), \end{aligned}$$

we have $u \in B^n(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet))$.

Lemma 18.4. *For any* $u \in C(\mathscr{A})(X^\bullet, Y^\bullet)$ *and* $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$ *the following hold*.
(1) $\mathrm{Hom}^\bullet(Z^\bullet, C(u)) \cong C(\mathrm{Hom}^\bullet(Z^\bullet, u))$.
(2) $\mathrm{Hom}^\bullet(C(u), Z^\bullet) \cong C_\rho(T^{-1}(\mathrm{Hom}^\bullet(u, Z^\bullet)))$ *(see Proposition 2.10)*.

Proof. (1) Let us identify $\mathrm{Hom}^n(Z^\bullet, -)$ with $\mathscr{A}^{\mathbb{Z}}(Z^\bullet, -) \circ T^n$ for all $n \in \mathbb{Z}$. Then for any ${}^t[f\ g] \in \mathrm{Hom}^n(Z^\bullet, C(u)) \cong \mathscr{A}^{\mathbb{Z}}(Z^\bullet, T^{n+1}(X^\bullet)) \oplus \mathscr{A}^{\mathbb{Z}}(Z^\bullet, T^n(Y^\bullet))$ we have

$$(-1)^{n+1} d^n_{\mathrm{Hom}^\bullet(Z^\bullet,\,C(u))}(\begin{bmatrix} f \\ g \end{bmatrix}) = T(\begin{bmatrix} f \\ g \end{bmatrix}) \circ d_Z - d_{T^nC(u)} \circ \begin{bmatrix} f \\ g \end{bmatrix}$$

$$= \begin{bmatrix} T(f) \\ T(g) \end{bmatrix} d_Z - \begin{bmatrix} d_{T^{n+1}X} & 0 \\ (-1)^n T^{n+1}(u) & d_{T^nY} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

$$= \begin{bmatrix} T(f) \circ d_Z - d_{T^{n+1}X} \circ f \\ (-1)^{n+1} T^{n+1}(u) \circ f + T(g) \circ d_Z - d_{T^nY} \circ g \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^{n+2} d^{n+1}_{\mathrm{Hom}^\bullet(Z^\bullet, X^\bullet)} & 0 \\ (-1)^{n+1}\mathrm{Hom}^{n+1}(Z^\bullet, u) & (-1)^{n+1} d^n_{\mathrm{Hom}^\bullet(Z^\bullet, Y^\bullet)} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}.$$

Thus we have

$$d^n_{\mathrm{Hom}^\bullet(Z^\bullet, C(u))} = \begin{bmatrix} -d^{n+1}_{\mathrm{Hom}^\bullet(Z^\bullet, X^\bullet)} & 0 \\ \mathrm{Hom}^{n+1}(Z^\bullet, u) & d^n_{\mathrm{Hom}^\bullet(Z^\bullet, Y^\bullet)} \end{bmatrix}.$$

(2) Let us identify $\mathrm{Hom}^n(-, Z^\bullet)$ with $\mathscr{A}^{\mathbb{Z}}(-, Z^\bullet) \circ T^{-n}$ for all $n \in \mathbb{Z}$. Then for any ${}^t[g \ \ f] \in \mathscr{A}^{\mathbb{Z}}(T^{-n}(Y^\bullet), Z^\bullet) \oplus \mathscr{A}^{\mathbb{Z}}(T^{-(n-1)}(X^\bullet), Z^\bullet) \cong \mathrm{Hom}^n(T^{-n}C(u), Z^\bullet)$, since ${}^t[g \ \ f]$ corresponds to a morphism $[f \ \ g] : T^{-(n-1)}(X^\bullet) \oplus T^{-n}(Y^\bullet) \to Z^\bullet$ in $\mathscr{A}^{\mathbb{Z}}$, we have

$$d^n_{\mathrm{Hom}^\bullet(C(u), Z^\bullet)}\left(\begin{bmatrix} g \\ f \end{bmatrix}\right)$$

$$= [f \ \ g] \circ d_{T^{-(n+1)}C(u)} + T^{-1}(d_Z \circ [f \ \ g])$$

$$= [f \ \ g] \circ \begin{bmatrix} d_{T^{-n}X} & 0 \\ (-1)^{n+1} T^{-n}(u) & d_{T^{-(n+1)}Y} \end{bmatrix} + [T^{-1}(d_Z \circ f) \ \ T^{-1}(d_Z \circ g)]$$

$$= [f \circ d_{T^{-n}X} + T^{-1}(d_Z \circ f) + (-1)^{n+1} g \circ T^{-n}(u) \quad g \circ d_{T^{-(n+1)}Y} + T^{-1}(d_Z \circ g)]$$

$$= [d^{n-1}_{\mathrm{Hom}^\bullet(X^\bullet, Z^\bullet)}(f) + (-1)^{n+1} g \circ T^{-n}(u) \quad d^n_{\mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet)}(g)]$$

$$= [d^{n-1}(f) + (-1)^{n-1} g \circ T^{-n}(u) \quad d^n(g)].$$

Thus the differential is given by

$$\begin{bmatrix} g \\ f \end{bmatrix} \mapsto \begin{bmatrix} d^n_{\mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet)} & 0 \\ (-1)^{n-1}\mathrm{Hom}^n(u, Z^\bullet) & d^{n-1}_{\mathrm{Hom}^\bullet(X^\bullet, Z^\bullet)} \end{bmatrix} \begin{bmatrix} g \\ f \end{bmatrix}$$

and we have

$$d^n_{\mathrm{Hom}^\bullet(C(u), Z^\bullet)} = \begin{bmatrix} d^n_{\mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet)} & 0 \\ (-1)^{n-1}\mathrm{Hom}^n(u, Z^\bullet) & d^{n-1}_{\mathrm{Hom}^\bullet(X^\bullet, Z^\bullet)} \end{bmatrix}.$$

Proposition 18.5. *The bifunctor* $\mathrm{Hom}^\bullet$ *gives rise to a bi-∂-functor*

$$\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z}).$$

Proof. For any $u \in \mathrm{Htp}(X^\bullet, Y^\bullet)$ in $C(\mathscr{A})$ and $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$, by Lemma 18.4 and Proposition 3.1 we have

$$\mathrm{Hom}^\bullet(Z^\bullet, u) \in \mathrm{Htp}(\mathrm{Hom}^\bullet(Z^\bullet, X^\bullet), \mathrm{Hom}^\bullet(Z^\bullet, Y^\bullet)),$$
$$\mathrm{Hom}^\bullet(u, Z^\bullet) \in \mathrm{Htp}(\mathrm{Hom}^\bullet(Y^\bullet, Z^\bullet), \mathrm{Hom}^\bullet(X^\bullet, Z^\bullet)).$$

Thus the bifunctor $\mathrm{Hom}^\bullet : C(\mathscr{A})^{\mathrm{op}} \times C(\mathscr{A}) \to C(\mathrm{Mod}\,\mathbb{Z})$ gives rise to a bifunctor

$$\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z}).$$

Next, by Lemmas 18.4 and 18.2

$$\mathrm{Hom}^\bullet(Z^\bullet, -) : K(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z})$$

is a ∂-functor for all $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. Also, by Lemmas 18.4 and 18.2 and Proposition 2.10

$$\mathrm{Hom}^\bullet(-, Z^\bullet) : K(\mathscr{A})^{\mathrm{op}} \to K(\mathrm{Mod}\,\mathbb{Z}).$$

is a ∂-functor for all $Z^\bullet \in \mathrm{Ob}(C(\mathscr{A}))$. Finally, it follows by Lemma 18.2 that

$$\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z})$$

is a bi-∂-functor.

Remark 18.1. (1) Let $X^\bullet \in \mathrm{Ob}(K^-(\mathscr{A}))$ and $Y^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$. Then $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) \in \mathrm{Ob}(K^+(\mathrm{Mod}\,\mathbb{Z}))$ and every $\mathrm{Hom}^n(X^\bullet, Y^\bullet)$ is a finite direct sum.

(2) Let $X^\bullet \in \mathrm{Ob}(K^+(\mathscr{A}))$ and $Y^\bullet \in \mathrm{Ob}(K^-(\mathscr{A}))$. Then $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet) \in \mathrm{Ob}(K^-(\mathrm{Mod}\,\mathbb{Z}))$ and every $\mathrm{Hom}^n(X^\bullet, Y^\bullet)$ is a finite direct sum.

(3) $\mathrm{Hom}^\bullet(X, Y^\bullet) \cong \mathscr{A}(X, Y^\bullet)$ for all $X \in \mathrm{Ob}(\mathscr{A})$ and $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$.

(4) $\mathrm{Hom}^\bullet(X^\bullet, Y) \cong \mathscr{A}(X^\bullet, Y)$ for all $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathscr{A})$.

(5) Let $\mathscr{B}$ be another abelian category and $F : \mathscr{A} \to \mathscr{B}$ an additive functor which has a left adjoint $G : \mathscr{B} \to \mathscr{A}$. Then we have a natural isomorphism

$$\mathrm{Hom}^\bullet(G(X^\bullet), Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet(X^\bullet, F(Y^\bullet))$$

for $X^\bullet \in \mathrm{Ob}(K(\mathscr{B}))$ and $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$.

Lemma 18.6. *The following hold.*

(1) *If either* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *or* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ *is acyclic, so is* $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$.

(2) *If eihter* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *or* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$ *is acyclic, so is* $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$.

(3) *If either* $P^\bullet \in \mathrm{Ob}(K^-(\mathscr{P}))$ *or* $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *is acyclic, so is* $\mathrm{Hom}^\bullet(P^\bullet, Y^\bullet)$.

(4) *If either* $P^\bullet \in \mathrm{Ob}(K(\mathscr{P})_{\mathrm{L}})$ *or* $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *is acyclic, so is* $\mathrm{Hom}^\bullet(P^\bullet, Y^\bullet)$.

Proof. (1) Note that $T^n(I^\bullet) \in \mathrm{Ob}(K^+(\mathscr{I}))$ for all $n \in \mathbb{Z}$. In case $X^\bullet \in \mathrm{Ob}(\mathscr{U})$, by Lemmas 18.3 and 4.4 $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ is acyclic. Assume $I^\bullet \in \mathrm{Ob}(\mathscr{U})$. Then by Lemma 4.4 $T^n(I^\bullet) = 0$ in $K(\mathscr{A})$ for all $n \in \mathbb{Z}$. Thus by Lemma 18.3 $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ is acyclic.

(2) Note that $T^n(I^\bullet) \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$ for all $n \in \mathbb{Z}$. In case $X^\bullet \in \mathrm{Ob}(\mathscr{U})$, then by Lemma 18.3 $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ is acyclic. Assume $I^\bullet \in \mathrm{Ob}(\mathscr{U})$. Then by Lemma 12.15 $T^n(I^\bullet) = 0$ in $K(\mathscr{A})$ for all $n \in \mathbb{Z}$. Thus by Lemma 18.3 $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ is acyclic.

(3) Dual of (1).

(4) Dual of (2).

Proposition 18.7. *Assume* $\mathscr{A}$ *has enough injectives. Then the following hold.*

(1) $\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K^+(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z})$ *has a right derived functor*

$$\boldsymbol{R}\,\mathrm{Hom}^\bullet = \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}\,\mathrm{Hom}^\bullet : D(\mathscr{A})^{\mathrm{op}} \times D^+(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$$

such that $\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, I^\bullet) \cong \mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ *provided* $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

(2) *If* $\mathscr{A}$ *satisfies the condition Ab4*, then* $\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K(\mathscr{A}) \to K(\mathrm{Mod}\,\mathbb{Z})$ *has a right derived functor*

$$\boldsymbol{R}\,\mathrm{Hom}^\bullet = \boldsymbol{R}_{\mathrm{I}}\boldsymbol{R}_{\mathrm{II}}\,\mathrm{Hom}^\bullet : D(\mathscr{A})^{\mathrm{op}} \times D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$$

such that $\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, I^\bullet) \cong \mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ *provided* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$.

Proof. (1) For any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$, since by Proposition 4.7 and Lemma 18.6(1) $K^+(\mathscr{I})$ satisfies the hypotheses of Proposition 13.6 for $\mathrm{Hom}^\bullet(X^\bullet, -) : K^+(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$. Thus by Lemma 17.1 we have a bi-∂-functor

$$\boldsymbol{R}_{\mathrm{II}}\,\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times D^+(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z}).$$

Then, for any $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$, by Lemma 18.6(1)

$$\boldsymbol{R}_{\mathrm{II}}\operatorname{Hom}^\bullet(-,\ I^\bullet) \cong Q \circ \operatorname{Hom}^\bullet(-,\ I^\bullet) : K(\mathcal{A}) \to D(\operatorname{Mod}\mathbb{Z})$$

vanishes on $\mathcal{U}$. Thus, since by Proposition 10.13 $K^+(\mathcal{I}) \xrightarrow{\sim} D^+(\mathcal{A})$, Proposition 17.3 applies.

(2) For any $X^\bullet \in \operatorname{Ob}(K(\mathcal{A}))$, since by Proposition 12.15 and Lemma 18.6(2) $K(\mathcal{I})_{\mathrm{L}}$ satisfies the hypotheses of Proposition 13.6 for $\operatorname{Hom}^\bullet(X^\bullet, -) : K(\mathcal{A}) \to D(\operatorname{Mod}\mathbb{Z})$. Thus by Lemma 17.1 we have a bi-∂-functor

$$\boldsymbol{R}_{\mathrm{II}}\operatorname{Hom}^\bullet : K(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\operatorname{Mod}\mathbb{Z}).$$

Then, for any $I^\bullet \in \operatorname{Ob}(K(\mathcal{I})_{\mathrm{L}})$, by Lemma 18.6(2)

$$\boldsymbol{R}_{\mathrm{II}}\operatorname{Hom}^\bullet(-,\ I^\bullet) \cong Q \circ \operatorname{Hom}^\bullet(-,\ I^\bullet) : K(\mathcal{A})^{\mathrm{op}} \to D(\operatorname{Mod}\mathbb{Z})$$

vanishes on $\mathcal{U}$. Thus, since by Proposition 12.16(1) $K(\mathcal{I})_{\mathrm{L}} \xrightarrow{\sim} D(\mathcal{A})$, Proposition 17.3 applies.

Proposition 18.8 (Dual of Proposition 18.7). *Assume $\mathcal{A}$ has enough projectives. Then the following hold*

(1) $\operatorname{Hom}^\bullet : K^-(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\operatorname{Mod}\mathbb{Z})$ *has a right derived functor*

$$\boldsymbol{R}\operatorname{Hom}^\bullet = \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}\operatorname{Hom}^\bullet : D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\operatorname{Mod}\mathbb{Z})$$

such that $\boldsymbol{R}\operatorname{Hom}^\bullet(P^\bullet,\ Y^\bullet) \cong \operatorname{Hom}^\bullet(P^\bullet,\ Y^\bullet)$ *provided* $P^\bullet \in \operatorname{Ob}(K^-(\mathcal{P}))$.

(2) *If $\mathcal{A}$ satisfies the condition Ab4, then* $\operatorname{Hom}^\bullet : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(\operatorname{Mod}\mathbb{Z})$ *has a right derived functor*

$$\boldsymbol{R}\operatorname{Hom}^\bullet = \boldsymbol{R}_{\mathrm{II}}\boldsymbol{R}_{\mathrm{I}}\operatorname{Hom}^\bullet : D(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\operatorname{Mod}\mathbb{Z})$$

such that $\boldsymbol{R}\operatorname{Hom}^\bullet(P^\bullet,\ Y^\bullet) \cong \operatorname{Hom}^\bullet(P^\bullet,\ Y^\bullet)$ *provided* $P^\bullet \in \operatorname{Ob}(K(\mathcal{P})_{\mathrm{L}})$.

Remark 18.2. (1) For any $P \in \mathcal{P}$, we have $Q \circ \mathcal{A}(P, -) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(P, -) \circ Q$.

(2) For any $I \in \mathcal{I}$, we have $Q \circ \mathcal{A}(-, I) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(-, I) \circ Q$.

Proposition 18.9. *Assume $\mathcal{A}$ has enough injectives. Then the following hold.*

(1) *For any* $X^\bullet \in \operatorname{Ob}(D(\mathcal{A}))$, $Y^\bullet \in \operatorname{Ob}(D^+(\mathcal{A}))$ *and* $i \in \mathbb{Z}$ *we have*

$$H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet,\ Y^\bullet)) \cong \operatorname{Ext}^i(X^\bullet,\ Y^\bullet).$$

(2) *If $\mathcal{A}$ satisfies the condition Ab4*, then for any* $X^\bullet \in \operatorname{Ob}(D(\mathcal{A}))$, $Y^\bullet \in \operatorname{Ob}(D(\mathcal{A}))$ *and i*

$\in \mathbb{Z}$ *we have*

$$H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) \cong \operatorname{Ext}^i(X^\bullet, Y^\bullet).$$

Proof. (1) By Proposition 4.7 we have a quasi-isomorphism $Y^\bullet \to I^\bullet$ with $I^\bullet \in \operatorname{Ob}(K^+(\mathscr{I}))$. Thus by Lemma 18. 3 and Proposition 10.12 we have

$$\begin{aligned} H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) &\cong H^i(\operatorname{Hom}^\bullet(X^\bullet, I^\bullet)) \\ &\cong K(\mathscr{A})(X^\bullet, T^i(I^\bullet)) \\ &\cong D(\mathscr{A})(X^\bullet, T^i(I^\bullet)) \\ &\cong D(\mathscr{A})(X^\bullet, T^i(Y^\bullet)) \\ &\cong \operatorname{Ext}^i(X^\bullet, Y^\bullet). \end{aligned}$$

(2) By Proposition 12.15 we have a quasi-isomorphism $Y^\bullet \to I^\bullet$ with $I^\bullet \in \operatorname{Ob}(K(\mathscr{I})_{\mathrm{L}})$. Thus by Lemma 18. 3 and Proposition 9.13(2) we have

$$\begin{aligned} H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) &\cong H^i(\operatorname{Hom}^\bullet(X^\bullet, I^\bullet)) \\ &\cong K(\mathscr{A})(X^\bullet, T^i(I^\bullet)) \\ &\cong D(\mathscr{A})(X^\bullet, T^i(I^\bullet)) \\ &\cong D(\mathscr{A})(X^\bullet, T^i(Y^\bullet)) \\ &\cong \operatorname{Ext}^i(X^\bullet, Y^\bullet). \end{aligned}$$

Proposition 18.10 (Dual of Proposition 18.9). *Assume* $\mathscr{A}$ *has enough projectives. Then the following hold.*

(1) *For any* $X^\bullet \in \operatorname{Ob}(D^-(\mathscr{A}))$, $Y^\bullet \in \operatorname{Ob}(D(\mathscr{A}))$ *and* $i \in \mathbb{Z}$ *we have*

$$H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) \cong \operatorname{Ext}^i(X^\bullet, Y^\bullet).$$

(2) *If* $\mathscr{A}$ *satisfies the condition Ab4, then for any* $X^\bullet \in \operatorname{Ob}(D(\mathscr{A}))$, $Y^\bullet \in \operatorname{Ob}(D(\mathscr{A}))$ *and* $i \in \mathbb{Z}$ *we have*

$$H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, Y^\bullet)) \cong \operatorname{Ext}^i(X^\bullet, Y^\bullet).$$

Proposition 18.11. *For any* $X^\bullet \in \operatorname{Ob}(C(\mathscr{A}))$ *with* $A = \operatorname{End}_{C(\mathscr{A})}(X^\bullet)$, *we have* ∂*-functors*

$$\operatorname{Hom}^\bullet(X^\bullet, -) : K(\mathscr{A}) \to K(\operatorname{Mod} A^{\mathrm{op}}), \qquad \operatorname{Hom}^\bullet(-, X^\bullet) : K(\mathscr{A})^{\mathrm{op}} \to K(\operatorname{Mod} A).$$

Furthermore, the following hold.

(1) *If* $\mathscr{A}$ *has enough injectives, then* $\operatorname{Hom}^\bullet(X^\bullet, -) : K^+(\mathscr{A}) \to K(\operatorname{Mod} A^{\mathrm{op}})$ *has a right*

derived functor $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, -)$.

(2) *If* $\mathscr{A}$ *has enough injectives and satisfies the condition Ab4*, then* $\operatorname{Hom}^\bullet(X^\bullet, -) : K(\mathscr{A}) \to K(\operatorname{Mod} A^{\mathrm{op}})$ *has a right derived functor* $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, -)$.

(2) *If* $\mathscr{A}$ *has enough projectives, then* $\operatorname{Hom}^\bullet(-, X^\bullet) : K^-(\mathscr{A})^{\mathrm{op}} \to K(\operatorname{Mod} A)$ *has a right derived functor* $\boldsymbol{R}\operatorname{Hom}^\bullet(-, X^\bullet)$.

(4) *If* $\mathscr{A}$ *has enough projectives and satisfies the condition Ab4, then* $\operatorname{Hom}^\bullet(-, X^\bullet) : K(\mathscr{A})^{\mathrm{op}} \to K(\operatorname{Mod} A)$ *has a right derived functor* $\boldsymbol{R}\operatorname{Hom}^\bullet(-, X^\bullet)$.

Proof. By Lemma 18.1 we have functors

$$\operatorname{Hom}^\bullet(X^\bullet, -) : C(\mathscr{A}) \to C(\operatorname{Mod} A^{\mathrm{op}}), \quad \operatorname{Hom}^\bullet(-, X^\bullet) : C(\mathscr{A})^{\mathrm{op}} \to C(\operatorname{Mod} A).$$

Thus by Proposition 18.5 we get ∂-functors

$$\operatorname{Hom}^\bullet(X^\bullet, -) : K(\mathscr{A}) \to K(\operatorname{Mod} A^{\mathrm{op}}), \quad \operatorname{Hom}^\bullet(-, X^\bullet) : K(\mathscr{A})^{\mathrm{op}} \to K(\operatorname{Mod} A).$$

The remaining assertions are immediate by the construction of $\boldsymbol{R}\operatorname{Hom}^\bullet$.

Throughout the rest of this section, R is a commutative ring and A, B are R-algebras.

Definition 18.3. For a ring A, we denote by Inj A, Proj A and Flat A the collection of injective, projective and flat left A-modules, respectively. Also, we denote by proj A the collection of finitely generated projective left A-modules. Right A-modules are considered as left A^{op}-modules, where A^{op} denotes the opposite ring of A.

Definition 18.4. For a ring A, we denote by mod A the full subcategory of Mod A consisting of finitely presented modules. In case A is left coherent (resp. noetherian), mod A consists of the finitely presented (resp. generated) modules and coinsides with the thick subcategory of Mod A consisting of coherent modules.

Definition 18.5. For * = +, –, b or nothing, we denote by $K_c^*(\operatorname{Mod} A)$ the full triangulated subcategory of $K^*(\operatorname{Mod} A)$ consistsing of complexes $X^\bullet \in \operatorname{Ob}(K^*(\operatorname{Mod} A))$ with the $H^n(X^\bullet)$ coherent and by $D_c^*(\operatorname{Mod} A)$ the corresponding derived category.

Definition 18.6. A left $A \otimes_R B^{\mathrm{op}}$-module V is an A-B-bimodule V such that the action of R on V via A coinsides with that of R on V via B, i.e., $rv = vr$ for all $r \in R$ and $v \in V$.

Remark 18.4. For $V^\bullet \in \operatorname{Ob}(C(\operatorname{Mod} A \otimes_R B^{\mathrm{op}}))$ the following hold.

(1) We have ring homomorphisms

$$\phi : A \to \operatorname{End}_{C(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet), \quad \psi : B \to \operatorname{End}_{C(\mathrm{Mod}\, A)}(V^\bullet)^{\mathrm{op}}$$

such that $\phi(a)^n(v) = av$ for all $a \in A$, $n \in \mathbb{Z}$ and $v \in V^n$, and $\psi(b)^n(v) = vb$ for all $b \in B$, $n \in \mathbb{Z}$ and $v \in V^n$.

(2) If $V^\bullet \in \mathrm{Ob}(K(\mathrm{Flat}\, B^{\mathrm{op}}))$ and $I^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\, A))$, then $\mathrm{Hom}^\bullet(V^\bullet, I^\bullet) \in \mathrm{Ob}(K(\mathrm{Inj}\, B))$.

(3) If $V^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\, B^{\mathrm{op}}))$ and $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A))$, then $\mathrm{Hom}^\bullet(P^\bullet, V^\bullet) \in \mathrm{Ob}(K(\mathrm{Inj}\, B^{\mathrm{op}}))$.

Proposition 18.12. (1) *We have a bi-∂-functor*

$$\mathrm{Hom}^\bullet : K(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}})^{\mathrm{op}} \times K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, B)$$

which has a right derived functor

$$\boldsymbol{R}\mathrm{Hom}^\bullet : D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}})^{\mathrm{op}} \times D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, B)$$

such that $\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, X^\bullet) \cong \mathrm{Hom}^\bullet(V^\bullet, X^\bullet)$ *provided either* $V^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A \otimes_R B^{\mathrm{op}})_{\mathrm{L}})$ *or* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\, A)_{\mathrm{L}})$.

(2) *We have a bi-∂-functor*

$$\mathrm{Hom}^\bullet : K(\mathrm{Mod}\, A)^{\mathrm{op}} \times K(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \to K(\mathrm{Mod}\, B^{\mathrm{op}})$$

which has a right derived functor

$$\boldsymbol{R}\mathrm{Hom}^\bullet : D(\mathrm{Mod}\, A)^{\mathrm{op}} \times D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \to D(\mathrm{Mod}\, B^{\mathrm{op}})$$

such that $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \cong \mathrm{Hom}^\bullet(X^\bullet, V^\bullet)$ *provided either* $X^\bullet \in \mathrm{Ob}(C(\mathrm{Proj}\, A)_{\mathrm{L}})$ *or* $V^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\, A \otimes_R B^{\mathrm{op}})_{\mathrm{L}})$.

Proof. Straightforward.

Remark 18.6. Let $B \to A$ be a homomorphism of R-algebras and $U : \mathrm{Mod}\, A \to \mathrm{Mod}\, B$ the induced functor. Then the extended ∂-functor $U : K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, B)$ has a right derived functor $\boldsymbol{R}\mathrm{Hom}^\bullet({}_AA_B, -)$ such that $Q \circ U \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet({}_AA_B, -) \circ Q$.

Proposition 18.13. *We have a bi-∂-functor*

$$\mathrm{Hom}^\bullet : K(\mathrm{Mod}\, A)^{\mathrm{op}} \times K(\mathrm{Mod}\, R) \to K(\mathrm{Mod}\, A^{\mathrm{op}})$$

which has a right derived functor

$$\boldsymbol{R}\operatorname{Hom}^\bullet : D(\operatorname{Mod} A)^{\mathrm{op}} \times D(\operatorname{Mod} R) \to D(\operatorname{Mod} A^{\mathrm{op}})$$

such that $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \cong \operatorname{Hom}^\bullet(X^\bullet, E^\bullet)$ *provided either* $X^\bullet \in \operatorname{Ob}(K(\operatorname{Proj} A)_{\mathrm{L}})$ *or* $E^\bullet \in \operatorname{Ob}(K(\operatorname{Inj} R)_{\mathrm{L}})$.

Proof. Straightforward.

§19. The left derived functor of $\otimes$

Throught this section, R is a commutative ring and A, B are R-algebras. For any ring A, we denote by $K(\mathrm{Inj}\ A)_{\mathrm{L}}$ (resp. $K(\mathrm{Proj}\ A)_{\mathrm{L}}$) the full subcategory of $K(\mathrm{Inj}\ A)$ (resp. $K(\mathrm{Proj}\ A)$) consisting of $\mathcal{U}$-local (resp. $\mathcal{U}$-coloca) complexes, where $\mathcal{U}$ is the épaisse subcategory of $K(\mathrm{Mod}\ A)$ consisting of acyclic complexes. Also, we denote by E an injective cogenerator in Mod R and by D both $\mathrm{Hom}_R(-, E)$ and $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, E)$.

Definition 19.1. For $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A))$, we define a double complex $C^{\bullet\bullet}$ in Mod R as follows:

$$C^{p,q} = M^p \otimes_A X^q,$$
$$d_1^{p,q} = d_M^p \otimes \mathrm{id}_{X^q},$$
$$d_2^{p,q} = (-1)^{p+q}\,\mathrm{id}_{M^p} \otimes d_X^q$$

for $p, q \in \mathbb{Z}$, and set $M^\bullet \otimes X^\bullet = t'(C^{\bullet\bullet})$. Then we get a bifunctor

$$\otimes : C(\mathrm{Mod}\ A^{\mathrm{op}}) \times C(\mathrm{Mod}\ A) \to C(\mathrm{Mod}\ R)$$

such that

$$[\,M^\bullet \otimes X^\bullet]^n = \bigoplus_{p+q=n} M^p \otimes_A X^q$$

for $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A))$ and $n \in \mathbb{Z}$, and

$$d^n_{M^\bullet \otimes X^\bullet}(m \otimes x) = d_M^p(m) \otimes x + (-1)^n m \otimes d_X^q(x)$$

for $n \in \mathbb{Z}$, $p, q \in \mathbb{Z}$ with $p + q = n$ and $m \otimes x \in M^p \otimes_A X^q$.

Definition 19.2. For any $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A))$ we set

$$M^\bullet \otimes_{\mathrm{gr}} X^\bullet = \bigoplus_{p+q=0} M^p \otimes_A X^q.$$

Then we get a bifunctor

$$\otimes_{\mathrm{gr}} : C(\mathrm{Mod}\ A^{\mathrm{op}}) \times C(\mathrm{Mod}\ A) \to \mathrm{Mod}\ R.$$

Lemma 19.1. *For any* $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\ A))$ *the following hold.*

(1) *In case we identify* $(M^\bullet \otimes X^\bullet)^n$ *with* $T^n M^\bullet \otimes_{\mathrm{gr}} X^\bullet$ *for all* $n \in \mathbb{Z}$, *the differntial is of the form*

$$d^n_{M^\bullet \otimes X^\bullet} = (-1)^n \{ d_{T^n M} \otimes_{\mathrm{gr}} \mathrm{id}_X + \mathrm{id}_{T^n M} \otimes_{\mathrm{gr}} d_X \}.$$

(2) *In case we identify* $(M^\bullet \otimes X^\bullet)^n$ *with* $M^\bullet \otimes_{\mathrm{gr}} T^n X^\bullet$ *for all* $n \in \mathbb{Z}$, *the differential is of the form*

$$d^n_{M^\bullet \otimes X^\bullet} = d_M \otimes_{\mathrm{gr}} \mathrm{id}_{T^n X} + \mathrm{id}_M \otimes_{\mathrm{gr}} d_{T^n X}.$$

Proof. Straightforward.

Definition 19.3. For any abelian category $\mathscr{A}$ we denote by $\rho : \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}^{\mathbb{Z}}}$ an automorphism of the identity functor $\mathbf{1}_{\mathscr{A}^{\mathbb{Z}}} : \mathscr{A}^{\mathbb{Z}} \to \mathscr{A}^{\mathbb{Z}}$ such that $\rho^n_X = (-1)^n \mathrm{id}_{X^n}$ for all $X \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ and $n \in \mathbb{Z}$.

Lemma 19.2. (1) *For any* $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A))$ *we have*

$$d_{TM^\bullet \otimes X^\bullet} = -T(d_{M^\bullet \otimes X^\bullet}), \quad d_{M^\bullet \otimes TX^\bullet} = T(d_{M^\bullet \otimes X^\bullet}).$$

(2) *We have isomorphisms of bifunctors*

$$\alpha : (- \otimes -) \circ (\mathbf{1} \times T) \xrightarrow{\sim} T \circ (- \otimes -), \quad \beta : (- \otimes -) \circ (T \times \mathbf{1}) \xrightarrow{\sim} T \circ (- \otimes -)$$

such that for any $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A))$,

$$\alpha_{(M,X)} = \rho_{M^\bullet \otimes TX^\bullet}, \quad \beta_{(M,X)} = \mathrm{id}_{TM^\bullet \otimes X^\bullet}.$$

In particular, $T\alpha \circ \beta_{(\mathbf{1} \times T)} + T\beta \circ \alpha_{(T \times \mathbf{1})} = 0$.

Proof. (1) Straightforward.
(2) Let $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A))$. Since

$$[M^\bullet \otimes TX^\bullet]^n = [M^\bullet \otimes X^\bullet]^{n+1} = [TM^\bullet \otimes X^\bullet]^n$$

for all $n \in \mathbb{Z}$, we have

$$M^\bullet \otimes TX^\bullet = T(M^\bullet \otimes X^\bullet) = TM^\bullet \otimes X^\bullet$$

in (Mod $R)^{\mathbb{Z}}$. Thus by the part (1) we have isomorphisms in $C(\text{Mod}\, R)$

$$\alpha_{(M,X)} = \rho_{M^\bullet \otimes TX^\bullet} : M^\bullet \otimes TX^\bullet \tilde{\to} T(M^\bullet \otimes X^\bullet),$$
$$\beta_{(M,X)} = \mathrm{id}_{TM^\bullet \otimes X^\bullet} : TM^\bullet \otimes X^\bullet \tilde{\to} T(M^\bullet \otimes X^\bullet).$$

It is obvious that $T\alpha \circ \beta_{(\mathbf{1}\times T)} + T\beta \circ \alpha_{(T\times \mathbf{1})} = 0$.

Lemma 19.3. (1) $M^\bullet \otimes C(u) \cong C_\rho(M^\bullet \otimes u)$ *for all* $M^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A^{\mathrm{op}}))$ *and a morphism* $u : X^\bullet \to Y^\bullet$ *in* $C(\text{Mod}\, A)$ *(see Proposition 2.10).*

(2) $C(u) \otimes X^\bullet \cong C(u \otimes X^\bullet)$ *for all* $X^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A))$ *and a morphism* $u : M^\bullet \to N^\bullet$ *in* $C(\text{Mod}\, A^{\mathrm{op}})$.

Proof. (1) We identify $(M^\bullet \otimes X^\bullet)^n$ with $M^\bullet \otimes_{\mathrm{gr}} T^n X^\bullet$ for all $M^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A))$ and $n \in \mathbb{Z}$. Let $M^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A^{\mathrm{op}}))$ and $u : X^\bullet \to Y^\bullet$ in $C(\text{Mod}\, A)$. Then

$$[M^\bullet \otimes C(u)]^n \cong (M^\bullet \otimes_{\mathrm{gr}} T^{n+1} X^\bullet) \oplus (M^\bullet \otimes_{\mathrm{gr}} T^n Y^\bullet),$$

$$\begin{aligned}
d^n_{M^\bullet \otimes C(u)} &= d_M \otimes_{\mathrm{gr}} \mathrm{id}_{T^n C(u)} + \mathrm{id}_M \otimes_{\mathrm{gr}} d_{T^n C(u)} \\
&= \begin{bmatrix} d_M \otimes_{\mathrm{gr}} \mathrm{id}_{T^{n+1}X} & 0 \\ 0 & d_M \otimes_{\mathrm{gr}} \mathrm{id}_{T^n Y} \end{bmatrix} + \begin{bmatrix} \mathrm{id}_M \otimes_{\mathrm{gr}} d_{T^{n+1}X} & 0 \\ (-1)^n \mathrm{id}_M \otimes_{\mathrm{gr}} T^{n+1}u & \mathrm{id}_M \otimes_{\mathrm{gr}} d_{T^n Y} \end{bmatrix} \\
&= \begin{bmatrix} d^{n+1}_{M^\bullet \otimes X^\bullet} & 0 \\ (-1)^n (M^\bullet \otimes u)^{n+1} & d^n_{M^\bullet \otimes Y^\bullet} \end{bmatrix}
\end{aligned}$$

for all $n \in \mathbb{Z}$.

(2) We identify $(M^\bullet \otimes X^\bullet)^n$ with $T^n M^\bullet \otimes_{\mathrm{gr}} X^\bullet$ for all $M^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A))$ and $n \in \mathbb{Z}$. Let $X^\bullet \in \mathrm{Ob}(C(\text{Mod}\, A))$ and $u : M^\bullet \to N^\bullet$ *in* $C(\text{Mod}\, A^{\mathrm{op}})$. Then, as in the part (1), we have

$$[C(u) \otimes X^\bullet]^n \cong (T^{n+1} M^\bullet \otimes_{\mathrm{gr}} X^\bullet) \oplus (T^n N^\bullet \otimes_{\mathrm{gr}} X^\bullet),$$

$$d^n_{C(u) \otimes X^\bullet} = \begin{bmatrix} -d^{n+1}_{M^\bullet \otimes X^\bullet} & 0 \\ (u \otimes X^\bullet)^{n+1} & d^n_{N^\bullet \otimes X^\bullet} \end{bmatrix}$$

for all $n \in \mathbb{Z}$.

Proposition 19.4. *The bifunctor* $\otimes$ *gives rise to a bi-∂-functor*

$$\otimes : K(\text{Mod } A^{\text{op}}) \times K(\text{Mod } A) \to K(\text{Mod } R).$$

Proof. By Lemma 19.3 and Proposition 3.1, for any $u \in \text{Htp}(X^\bullet, Y^\bullet)$ in $C(\text{Mod } A)$ and $M^\bullet \in \text{Ob}(C(\text{Mod } A^{\text{op}}))$ we have

$$M^\bullet \otimes u \in \text{Htp}(M^\bullet \otimes X^\bullet, M^\bullet \otimes Y^\bullet),$$

and for any $u \in \text{Htp}(M^\bullet, N^\bullet)$ in $C(\text{Mod } A^{\text{op}})$ and $X^\bullet \in \text{Ob}(C(\text{Mod } A))$ we have

$$u \otimes X^\bullet \in \text{Htp}(M^\bullet \otimes X^\bullet, N^\bullet \otimes X^\bullet).$$

Thus the bifunctor $\otimes : C(\text{Mod } A^{\text{op}}) \times C(\text{Mod } A) \to C(\text{Mod } R)$ gives rise to a bifunctor

$$\otimes : K(\text{Mod } A^{\text{op}}) \times K(\text{Mod } A) \to K(\text{Mod } R).$$

Next, by Lemmas 19.3 and 19.2

$$- \otimes X^\bullet : K(\text{Mod } A^{\text{op}}) \to K(\text{Mod } R)$$

is a ∂-functor for all $X^\bullet \in \text{Ob}(K(\text{Mod } A))$. Also, by Lemmas 19.3, 19.2 and Proposition 2.10

$$M^\bullet \otimes - : K(\text{Mod } A) \to K(\text{Mod } R)$$

is a ∂-functor for all $M^\bullet \in \text{Ob}(K(\text{Mod } A^{\text{op}}))$. Finally, it follows by Lemma 19.2 that

$$\otimes : K(\text{Mod } A^{\text{op}}) \times K(\text{Mod } A) \to K(\text{Mod } R).$$

is a bi-∂-functor.

Remark 19.1. (1) $M^\bullet \otimes X^\bullet \in \text{Ob}(K^*(\text{Mod } R))$ for all $M^\bullet \in \text{Ob}(K^*(\text{Mod } A^{\text{op}}))$ and $X^\bullet \in \text{Ob}(K^*(\text{Mod } A))$, where $* = +, -$ or b.

(2) $M \otimes X^\bullet \cong M \otimes_A X^\bullet$ for all $M \in \text{Mod } A^{\text{op}}$ and $X^\bullet \in \text{Ob}(K(\text{Mod } A))$.

(3) $M^\bullet \otimes X \cong M^\bullet \otimes_A X$ for all $M^\bullet \in \text{Ob}(K(\text{Mod } A^{\text{op}}))$ and $X \in \text{Mod } A$.

Lemma 19.5. (1) *There exists a natural isomorphism*

$$\text{Hom}^\bullet(M^\bullet \otimes V^\bullet, N^\bullet) \xrightarrow{\sim} \text{Hom}^\bullet(M^\bullet, \text{Hom}^\bullet(V^\bullet, N^\bullet))$$

for $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $N^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, B^{\mathrm{op}}))$.

(2) *There exists a natural isomorphism*

$$\mathrm{Hom}^\bullet(V^\bullet \otimes X^\bullet, Y^\bullet) \tilde{\to} \mathrm{Hom}^\bullet(X^\bullet, \mathrm{Hom}^\bullet(V^\bullet, Y^\bullet))$$

for $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, B))$ *and* $Y^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A))$.

Proof. (1) For any $n \in \mathbb{Z}$, we may consider that

$$\mathrm{Hom}^n(M^\bullet \otimes V^\bullet, N^\bullet) = \prod_{p+q+r=n} \mathrm{Hom}_B(M^{-p} \otimes_A V^{-q}, N^r),$$

$$\mathrm{Hom}^n(M^\bullet, \mathrm{Hom}^\bullet(V^\bullet, N^\bullet)) = \prod_{p+q+r=n} \mathrm{Hom}_A(M^{-p}, \mathrm{Hom}_B(V^{-q}, N^r)).$$

For any $p, q, r \in \mathbb{Z}$ we have a natural isomorphism

$$\phi^{p,q,r} : \mathrm{Hom}_B(M^{-p} \otimes_A V^{-q}, N^r) \tilde{\to} \mathrm{Hom}_A(M^{-p}, \mathrm{Hom}_B(V^{-q}, N^r))$$

such that

$$\phi^{p,q,r}(u^{p,q,r})(m^p)(v^q) = (-1)^{\frac{q(q+1)}{2}} u^{p,q,r}(m^p \otimes v^q)$$

for $u^{p,q,r} \in \mathrm{Hom}_B(M^{-p} \otimes_A V^{-q}, N^r)$, $m^p \in M^{-p}$ and $v^q \in V^{-q}$. Thus for any $n \in \mathbb{Z}$ we have a natural isomorphism

$$\phi^n = (\phi^{p,q,r}) : \prod_{p+q+r=n} \mathrm{Hom}_B(M^{-p} \otimes_A V^{-q}, N^r) \tilde{\to} \prod_{p+q+r=n} \mathrm{Hom}_A(M^{-p}, \mathrm{Hom}_B(V^{-q}, N^r)).$$

It is easy to see that ϕ commutes with differentials.

(2) By symmetry.

Remark 19.2. For $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ the following hold.

(1) If $V^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, B^{\mathrm{op}}))$ and $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}}))$, then $P^\bullet \otimes V^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, B^{\mathrm{op}}))$.

(2) If $V^\bullet \in \mathrm{Ob}(K(\mathrm{Flat}\, B^{\mathrm{op}}))$ and $P^\bullet \in \mathrm{Ob}(K(\mathrm{Flat}\, A^{\mathrm{op}}))$, then $P^\bullet \otimes V^\bullet \in \mathrm{Ob}(K(\mathrm{Flat}\, B^{\mathrm{op}}))$.

Lemma 19.6. *For a bi-∂-functor* $\otimes : K(\mathrm{Mod}\, A^{\mathrm{op}}) \times K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, R)$ *the following hold.*

(1) *If either* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ *or* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$ *is acyclic, so is* $M^\bullet \otimes P^\bullet$.

(2) *If either* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ *or* $P^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A))$ *is acyclic, so is* $M^\bullet \otimes P^\bullet$.

(3) *If either* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}})_{\mathrm{L}})$ *or* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ *is acyclic, so is* $P^\bullet \otimes X^\bullet$.

(4) *If either* $P^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A^{\mathrm{op}}))$ *or* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ *is acyclic, so is* $P^\bullet \otimes X^\bullet$.

Proof. (1) Since by Lemma 19.5(2) $D(M^\bullet \otimes P^\bullet) \tilde{\to} \mathrm{Hom}^\bullet(P^\bullet, D(M^\bullet))$, Lemma 18.6(4) applies.

(2) Since by Lemma 19.5(1) $D(M^\bullet \otimes X^\bullet) \tilde{\to} \mathrm{Hom}^\bullet(M^\bullet, D(X^\bullet))$ with $D(X^\bullet) \in \mathrm{Ob}(K^+(\mathrm{Inj}\, A^{\mathrm{op}}))$, Lemma 18.6(1) applies.

(3) and (4) follow by symmetry.

Proposition 19.7. *The bi-∂-functor* $\otimes : K(\mathrm{Mod}\, A^{\mathrm{op}}) \times K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, R)$ *has both a left derived functor*

$$\overset{\boldsymbol{L}}{\otimes} = \overset{\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}}{\otimes} : D(\mathrm{Mod}\, A^{\mathrm{op}}) \times D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, R)$$

such that $M^\bullet \overset{\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}}{\otimes} P^\bullet \cong M^\bullet \otimes P^\bullet$ *for all* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$, *and a left derived functor*

$$\overset{\boldsymbol{L}}{\otimes} = \overset{\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}}{\otimes} : D(\mathrm{Mod}\, A^{\mathrm{op}}) \times D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, R)$$

such that $P^\bullet \overset{\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}}{\otimes} X^\bullet \cong P^\bullet \otimes X^\bullet$ *for all* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}})_{\mathrm{L}})$ *and* $X^\bullet \in K(\mathrm{Mod}\, A)$.

Proof. We claim first that $\overset{\boldsymbol{L}_{\mathrm{I}}\boldsymbol{L}_{\mathrm{II}}}{\otimes}$ exists. For any $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$, by Proposition 12.20 and Lemma 19.6(1) $K(\mathrm{Proj}\, A)_{\mathrm{L}}$ satisfies the hypotheses of Proposition 14.6 for $M^\bullet \otimes -$ $: K(\mathrm{Mod}\, A) \to K(\mathrm{Mod} R)$. Thus by Lemma 17.5 we have a bi-∂-functor

$$\overset{\boldsymbol{L}_{\mathrm{II}}}{\otimes} : K(\mathrm{Mod}\, A^{\mathrm{op}}) \times D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, R).$$

Then, for any $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$, by Lemma 19.6(2) $- \overset{\boldsymbol{L}_{\mathrm{II}}}{\otimes} P^\bullet : K(\mathrm{Mod}\, A^{\mathrm{op}}) \to D(\mathrm{Mod}\, R)$ vanishes on the acyclic complexes. Also, by Proposition 12.21(1) $K(\mathrm{Proj}\, A)_{\mathrm{L}} \tilde{\to} D(\mathrm{Mod}\, A)$. Thus Proposition 17.7 applies. The existence of $\overset{\boldsymbol{L}_{\mathrm{II}}\boldsymbol{L}_{\mathrm{I}}}{\otimes}$ follows by symmetry.

Proposition 19.8. $M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet \cong M^\bullet \otimes X^\bullet$ *provided either* $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A))$ *or* $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A^{\mathrm{op}}))$.

Proof. Assume $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A))$. By Proposition 4.11 there exists a quasi-isomorphism $s : P^\bullet \to X^\bullet$ with $P^\bullet \in \mathrm{Ob}(K^-(\mathrm{Proj}\, A))$. Then, since $C(s)$ is acyclic, and since $C(s) \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A))$, by Lemmas 19.3(1) and 19.6(1) $M^\bullet \otimes s$ is a quasi-isomorphism. Thus

$$\begin{aligned} M^\bullet \otimes X^\bullet &\cong M^\bullet \otimes P^\bullet \\ &\cong M^\bullet \overset{L}{\otimes} P^\bullet \\ &\cong M^\bullet \overset{L}{\otimes} X^\bullet. \end{aligned}$$

By symmetry, $M^\bullet \overset{L}{\otimes} X^\bullet \cong M^\bullet \otimes X^\bullet$ if $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\, A^{\mathrm{op}}))$.

Remark 19.3. $(F \overset{L}{\otimes} -) \circ Q \xrightarrow{\sim} Q \circ (F \otimes_A -)$ for all $F \in \mathrm{Flat}\, A^{\mathrm{op}}$.

Proposition 19.9. (1) *We have a bi-∂-functor*

$$\otimes : K(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \times K(\mathrm{Mod}\, B) \to K(\mathrm{Mod}\, A)$$

which has a left derived functor

$$\overset{L}{\otimes} : D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \times D(\mathrm{Mod}\, B) \to D(\mathrm{Mod}\, A)$$

such that $V^\bullet \overset{L}{\otimes} X^\bullet \cong V^\bullet \otimes X^\bullet$ *provided either* $V^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A \otimes_R B^{\mathrm{op}})_{\mathrm{L}})$ *or* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, B)_{\mathrm{L}})$.

(2) *We have a bi-∂-functor*

$$\otimes : K(\mathrm{Mod}\, A^{\mathrm{op}}) \times K(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \to K(\mathrm{Mod}\, B^{\mathrm{op}})$$

which has a left derived functor

$$\overset{L}{\otimes} : D(\mathrm{Mod}\, A^{\mathrm{op}}) \times D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}) \to D(\mathrm{Mod}\, B^{\mathrm{op}})$$

such that $M^\bullet \overset{L}{\otimes} V^\bullet \cong M^\bullet \otimes V^\bullet$ *provided either* $V^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A \otimes_R B^{\mathrm{op}})_{\mathrm{L}})$ *or* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}})_{\mathrm{L}})$.

Proof. Straightforward.

Remark 19.4. Let $B \to A$ be a homomorphism of R-algebras and $U : \mathrm{Mod}\, A \to \mathrm{Mod}\, B$ the induced functor. Then the extended ∂-functor $U : K(\mathrm{Mod}\, A) \to K(\mathrm{Mod}\, B)$ has a left derived functor $({}_BA_A \overset{L}{\otimes} -)$ such that $({}_BA_A \overset{L}{\otimes} -) \circ Q \xrightarrow{\sim} Q \circ U$.

Lemma 19.10. (1) *Assume A is commutative. Then there exists a natural isomorphism*

$$M^\bullet \otimes X^\bullet \xrightarrow{\sim} X^\bullet \otimes M^\bullet$$

for $M^\bullet,\ X^\bullet \in \operatorname{Ob}(C(\operatorname{Mod} A))$.

(2) *There exists a natural isomorphism*

$$M^\bullet \otimes (V^\bullet \otimes X^\bullet) \xrightarrow{\sim} (M^\bullet \otimes V^\bullet) \otimes X^\bullet.$$

for $M^\bullet \in \operatorname{Ob}(C(\operatorname{Mod} A^{\mathrm{op}}))$, $V^\bullet \in \operatorname{Ob}(C(\operatorname{Mod} A \otimes_R B^{\mathrm{op}}))$ *and* $X^\bullet \in \operatorname{Ob}(C(\operatorname{Mod} B))$.

Proof. (1) For any $n \in \mathbb{Z}$ we have a natural isomorphism

$$\phi^n : \bigoplus_{p+q=n} M^p \otimes_A X^q \xrightarrow{\sim} \bigoplus_{p+q=n} X^q \otimes_A M^p$$

such that

$$\phi^n(m^p \otimes x^q) = (-1)^{\frac{(p+q)(p+q-1)}{2}} x^q \otimes m^p$$

for $m^p \otimes x^q \in M^p \otimes_A X^q$, where $p, q \in \mathbb{Z}$ with $p + q = n$. It is easy to see that ϕ commutes with differentials.

(2) For any $n \in \mathbb{Z}$ we have a natural isomorphism

$$\phi^n : \bigoplus_{p+q+r=n} (M^p \otimes_A V^q) \otimes_B X^r \xrightarrow{\sim} \bigoplus_{p+q+r=n} M^p \otimes_A (V^q \otimes_B X^r)$$

such that

$$\phi^n((m^p \otimes v^q) \otimes x^r) = (-1)^{\frac{r(2q+r-1)}{2}} m^p \otimes (v^q \otimes x^r)$$

for $(m^p \otimes v^q) \otimes x^r \in (M^p \otimes_A V^q) \otimes_B X^r$, where $p, q, r \in \mathbb{Z}$ with $p + q + r = n$. It is easy to see that ϕ commutes with differentials.

Proposition 19.11. (1) *Assume A is commutative. Then there exists a natural isomorphism*

$$M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet \xrightarrow{\sim} X^\bullet \overset{\boldsymbol{L}}{\otimes} M^\bullet$$

for $M^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A))$ *and* $X^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A))$.

(2) *There exists a natural isomorphism*

$$M^\bullet \overset{L}{\otimes} (V^\bullet \overset{L}{\otimes} X^\bullet) \tilde{\to} (M^\bullet \overset{L}{\otimes} V^\bullet) \overset{L}{\otimes} X^\bullet$$

for $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A^{\mathrm{op}}))$, $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, B))$.

Proof. (1) Let $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ and $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$. By Proposition 12.20 we may assume $M^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$. Thus by Lemma 19.10(1) we have

$$\begin{aligned} M^\bullet \overset{L}{\otimes} X^\bullet &\cong M^\bullet \otimes X^\bullet \\ &\cong X^\bullet \otimes M^\bullet \\ &\cong X^\bullet \overset{L}{\otimes} M^\bullet. \end{aligned}$$

(2) Similar to (1).

§20. Hyper Tor

Throught this section, R is a commutative ring and A, B are R-algebras. For any ring A, we denote by $K(\text{Inj } A)_{\text{L}}$ (resp. $K(\text{Proj } A)_{\text{L}}$) the full subcategory of $K(\text{Inj } A)$ (resp. $K(\text{Proj } A)$) consisting of $\mathcal{U}$-local (resp. $\mathcal{U}$-colocal) complexes, where $\mathcal{U}$ is the épaisse subcategory of $K(\text{Mod } A)$ consisting of acyclic complexes. Also, we denote by E an injective cogenerator in Mod R and by D both $\text{Hom}_R(-, E)$ and $\boldsymbol{R}\text{Hom}^\bullet(-, E)$.

Definition 20.1. For $M^\bullet \in \text{Ob}(D(\text{Mod } A^{\text{op}}))$, $X^\bullet \in \text{Ob}(D(\text{Mod } A))$ and $n \in \mathbb{Z}$ we set

$$\text{Tor}_n(M^\bullet, X^\bullet) = H^{-n}(M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet),$$

which is called the n^{th} hyper Tor.

Proposition 20.1. *For any* $M^\bullet \in \text{Ob}(D(\text{Mod } A^{\text{op}}))$, $X^\bullet \in \text{Ob}(D(\text{Mod } A))$ *and* $i \in \mathbb{Z}$, *there exist isomorphisms*

$$D(\text{Tor}_i(M^\bullet, X^\bullet)) \cong \text{Ext}^i(M^\bullet, D(X^\bullet)),$$
$$D(\text{Tor}_i(M^\bullet, X^\bullet)) \cong \text{Ext}^i(X^\bullet, D(M^\bullet)).$$

Proof. Since by Proposition 12.20 we have a quasi-isomorphism $P^\bullet \to M^\bullet$ with $P^\bullet \in \text{Ob}(K(\text{Proj } A^{\text{op}})_{\text{L}})$, we have

$$\begin{aligned} D(\text{Tor}_i(M^\bullet, X^\bullet)) &\cong D(H^{-i}(M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet)) \\ &\cong D(H^{-i}(P^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet)) \\ &\cong D(H^{-i}(P^\bullet \otimes X^\bullet)) \\ &\cong H^i(D(P^\bullet \otimes X^\bullet)) \\ &\cong H^i(\text{Hom}^\bullet(P^\bullet, D(X^\bullet)) \\ &\cong H^i(\boldsymbol{R}\text{Hom}^\bullet(P^\bullet, D(X^\bullet)) \\ &\cong H^i(\boldsymbol{R}\text{Hom}^\bullet(M^\bullet, D(X^\bullet)) \\ &\cong \text{Ext}^i(M^\bullet, D(X^\bullet)). \end{aligned}$$

By symmetry, the last isomorphism follows.

Proposition 20.2. *For* $X^\bullet \in \text{Ob}(D(\text{Mod } A))$ *the following are equivalent.*

(1) $X^\bullet \in \text{Ob}(D(\text{Mod } A)_{\text{fTd}})$.

(2) $D(X^\bullet) \in \text{Ob}(D(\text{Mod } A^{\text{op}})_{\text{fid}})$.

Proof. By Proposition 20.1.

Proposition 20.3. *Let* $V^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$. *Then the following hold.*

(1) *For each exact sequence* $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ *in* $C(\mathrm{Mod}\, B)$, *we have a long exact sequence in* $\mathrm{Mod}\, A$

$$\cdots \to \mathrm{Tor}_i(V^\bullet, X^\bullet) \to \mathrm{Tor}_i(V^\bullet, Y^\bullet) \to \mathrm{Tor}_i(V^\bullet, Z^\bullet) \to \mathrm{Tor}_{i-1}(V^\bullet, X^\bullet) \to \cdots,$$

(2) *For each exact sequence* $0 \to L^\bullet \to M^\bullet \to N^\bullet \to 0$ *in* $C(\mathrm{Mod}\, A^{\mathrm{op}})$, *we have a long exact sequence in* $\mathrm{Mod}\, B^{\mathrm{op}}$

$$\cdots \to \mathrm{Tor}_i(L^\bullet, V^\bullet) \to \mathrm{Tor}_i(M^\bullet, V^\bullet) \to \mathrm{Tor}_i(N^\bullet, V^\bullet) \to \mathrm{Tor}_{i-1}(L^\bullet, V^\bullet) \to \cdots.$$

Proof. (1) Since $V^\bullet \overset{L}{\otimes} - : D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, B)$ is a ∂-functor, and since by Proposition 11.1 we have a triangle in $D(\mathrm{Mod}\, A)$ of the form $(X^\bullet, Y^\bullet, Z^\bullet, \cdot, \cdot, \cdot)$, we get a triangle in $\mathrm{Mod}\, B$ of the form

$$(V^\bullet \overset{L}{\otimes} X^\bullet, V^\bullet \overset{L}{\otimes} Y^\bullet, V^\bullet \overset{L}{\otimes} Z^\bullet, \cdot, \cdot, \cdot).$$

(2) By symmetry.

Proposition 20.4. $\mathrm{Tor}_i(M, X)$ *coinsides with the usual* $\mathrm{Tor}_i^A(M, X)$ *for all* $M \in \mathrm{Mod}\, A^{\mathrm{op}}$, $X \in \mathrm{Mod}\, A$ *and* $i \geq 0$.

Proof. Let $M \in \mathrm{Mod}\, A^{\mathrm{op}}$ and $X \in \mathrm{Mod}\, A$. Put $G = M \otimes_A - : \mathrm{Mod}\, A \to \mathrm{Mod}\, R$ and take a projective resolution $P^\bullet \to X$. Then for any $i \geq 0$ we have

$$\begin{aligned}\mathrm{Tor}_i(M, X) &\cong H^{-i}(M \overset{L}{\otimes} X) \\ &\cong H^{-i}(M \overset{L}{\otimes} P^\bullet) \\ &\cong H^{-i}(M \otimes P^\bullet) \\ &\cong H^{-i}(G(P^\bullet)) \\ &\cong \mathrm{Tor}_i^A(M, X).\end{aligned}$$

Definition 20.2. A complex $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ is said to have finite flat dimension on $\mathrm{Mod}\, A^{\mathrm{op}}$ if, for $i \gg 0$, $\mathrm{Tor}_i(-, X^\bullet)$ vanishes on $\mathrm{Mod}\, A^{\mathrm{op}}$. Sometimes, flat dimension is called Tor dimension. For $* = +, -$, b or nothing, we denote by $K^*(\mathrm{Mod}\, A)_{\mathrm{fTd}}$ the full subcategory of $K^*(\mathrm{Mod}\, A)$ consisting of $X^\bullet \in \mathrm{Ob}(K^*(\mathrm{Mod}\, A))$ which have finite flat dimension on $\mathrm{Mod}\, A^{\mathrm{op}}$.

Lemma 20.5. *For* $* = +, -, \mathrm{b}$ *or nothing, the following hold.*

(1) $K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}}$ *is a full triangulated subcategory of* $K^*(\mathrm{Mod}\ A)$.

(2) $\mathcal{U} \cap K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}}$ *is an épaisse subcategory of* $K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}}$, *where* $\mathcal{U}$ *is the épaisse subcategory of* $K(\mathrm{Mod}\ A)$ *consisting of acyclic complexes*.

Proof. (1) Let $X^\bullet \in \mathrm{Ob}(K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}})$ and $n \in \mathbb{Z}$. Then $\mathrm{Tor}_i(-, T^n(X^\bullet)) \cong \mathrm{Tor}_{i-n}(-, X^\bullet)$ vanishes on $\mathrm{Mod}\ A^{\mathrm{op}}$ for $i \gg 0$ and $T^n(X^\bullet) \in \mathrm{Ob}(K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}})$. Also, for any $u : X^\bullet \to Y^\bullet$ in $K^*(\mathrm{Mod}\ A)$ with $X^\bullet, Y^\bullet \in \mathrm{Ob}(K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}})$, since by Proposition 20.3 we have an exact sequence

$$\cdots \to \mathrm{Tor}_i(-, Y^\bullet) \to \mathrm{Tor}_i(-, C(u)) \to \mathrm{Tor}_{i+1}(-, X^\bullet) \to \cdots ,$$

$\mathrm{Tor}_i(-, C(u))$ vanishes on $\mathrm{Mod}\ A^{\mathrm{op}}$ for $i \gg 0$ and $C(u) \in \mathrm{Ob}(K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}})$.

(2) By Proposition 7.7.

Definition 20.3. For $* = +, -, \mathrm{b}$ or nothing, according to Lemma 20.5, we have a derived category

$$D^*(\mathrm{Mod}\ A)_{\mathrm{fTd}} = K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}} / \mathcal{U} \cap K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}},$$

where $\mathcal{U}$ is the épaisse subcategory of $K(\mathrm{Mod}\ A)$ consisting of acyclic complexes.

Proposition 20.6. *For* $* = +, -, \mathrm{b}$ *or nothing, the following hold.*

(1) $D^*(\mathrm{Mod}\ A)_{\mathrm{fpd}} \subset D^*(\mathrm{Mod}\ A)_{\mathrm{fTd}}$.

(2) *The canonical functor* $D^*(\mathrm{Mod}\ A)_{\mathrm{fTd}} \to D(\mathrm{Mod}\ A)$ *is fully faithful*.

Proof. (1) By Proposition 20.1.

(2) It follows by definition that $K^*(\mathrm{Mod}\ A)_{\mathrm{fTd}}$ is closed under quasi-isomorphism classes in $K^*(\mathrm{Mod}\ A)$. Thus by Proposition 8.17 the canonical functor $D^*(\mathrm{Mod}\ A)_{\mathrm{fTd}} \to D^*(\mathrm{Mod}\ A)$ is fully faithful.

Lemma 20.7. *For* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A)_{\mathrm{fTd}})$.

(2) *For* $i \gg 0$, $\mathrm{Tor}_i(-, X^\bullet)$ *vanishes on* $\mathrm{mod}\ A^{\mathrm{op}}$.

(3) *There exists an isomorphism* $P^\bullet \to X^\bullet$ *in* $D(\mathrm{Mod}\ A)$ *with* $P^\bullet \in \mathrm{Ob}(K^+(\mathrm{Flat}\ A))$.

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Take $n \in \mathbb{Z}$ such that, for $i > n$, $\mathrm{Tor}_i(-, X^\bullet)$ vanishes on the finitely presented modules. By Proposition 12.20 there exists a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet \in$

$\mathrm{Ob}(K(\mathrm{Proj}\,A)_{\mathrm{L}})$. Then for $i > n$ we have

$$\begin{aligned} H^{-i}(P^\bullet) &\cong H^{-i}(X^\bullet) \\ &\cong H^{-i}(A \overset{L}{\otimes} X^\bullet) \\ &\cong \mathrm{Tor}_i(A,\ X^\bullet) \\ &= 0. \end{aligned}$$

Thus by Lemma 10.6 we have a quasi-isomorphism $P^\bullet \to \sigma_{\geq -n}(P^\bullet)$. Next, since we have a projective resolution of $B^{-n}(P^\bullet)$

$$\cdots \to P^{-n-2} \to P^{-n-1} \to B^{-n}(P^\bullet) \to 0,$$

by Proposition 20.4 we have

$$\begin{aligned} \mathrm{Tor}_1(M,\ B^{-n}(P^\bullet)) &\cong H^{-(n+2)}((M \otimes_A P^\bullet)) \\ &\cong H^{-(n+2)}((M \overset{L}{\otimes} P^\bullet)) \\ &\cong H^{-(n+2)}((M \overset{L}{\otimes} X^\bullet)) \\ &\cong \mathrm{Tor}_{n+2}(M,\ X^\bullet) \\ &= 0 \end{aligned}$$

for all $M \in \mathrm{mod}\,A^{\mathrm{op}}$. Thus $B^{-n}(P^\bullet)$ is flat and $\sigma_{\geq -n}(P^\bullet) \in \mathrm{Ob}(K^+(\mathrm{Flat}\,A))$.

(3) $\Rightarrow$ (1). For $i \gg 0$, since $P^\bullet \in \mathrm{Ob}(K^+(\mathrm{Flat}\,A))$, we have

$$\begin{aligned} \mathrm{Tor}_i(M,\ X^\bullet) &\cong \mathrm{Tor}_i(M,\ P^\bullet) \\ &\cong H^{-i}((M \overset{L}{\otimes} P^\bullet)) \\ &\cong H^{-i}((M \otimes P^\bullet)) \\ &= 0 \end{aligned}$$

for all $M \in \mathrm{Mod}\,A^{\mathrm{op}}$.

Lemma 20.8. *For* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\,A))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\,A)_{\mathrm{fid}})$.

(2) *For* $i \gg 0$, $\mathrm{Ext}^i(-,\ X^\bullet)$ *vanishes on the finitely generated modules.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Take $n \in \mathbb{Z}$ such that, for $i > n$, $\mathrm{Ext}^i(-,\ X^\bullet)$ vanishes on the finitely generated modules. By Proposition 12.15(1) there exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\,A)_{\mathrm{L}})$. Then for $i > n$ we have

$$
\begin{aligned}
H^i(X^\bullet) &\cong H^i(\operatorname{Hom}^\bullet(A,\ X^\bullet))\\
&\cong H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(A,\ X^\bullet))\\
&\cong \operatorname{Ext}^i(A,\ X^\bullet)\\
&= 0.
\end{aligned}
$$

Thus by Lemma 10.7 we have a quasi-isomorphism $\sigma_n(I^\bullet) \to I^\bullet$. Next, since we have an injective resolution of $Z^n(I^\bullet)$

$$
0 \to Z^n(I^\bullet) \to I^n \to I^{n+1} \to \cdots ,
$$

for any finitely generated modules $Y \in \operatorname{Mod} A$ we have

$$
\begin{aligned}
\operatorname{Ext}^1(Y, Z^n(I^\bullet)) &\cong H^{n+1}(\operatorname{Hom}_A(Y,\ I^\bullet))\\
&\cong H^{n+1}(\boldsymbol{R}\operatorname{Hom}^\bullet(Y,\ I^\bullet))\\
&\cong H^{n+1}(\boldsymbol{R}\operatorname{Hom}^\bullet(Y,\ X^\bullet))\\
&\cong \operatorname{Ext}^{n+1}(Y,\ X^\bullet)\\
&= 0.
\end{aligned}
$$

It follows by Baer's criterion that $Z^n(I^\bullet)$ is injective. Thus $\sigma_n(I^\bullet) \in \operatorname{Ob}(K^-(\operatorname{Inj} A))$ and

$$
\begin{aligned}
\operatorname{Ext}^i(Y,\ X^\bullet) &\cong \operatorname{Ext}^i(Y, \sigma_n(I^\bullet))\\
&\cong H^i((\boldsymbol{R}\operatorname{Hom}^\bullet(Y, \sigma_n(I^\bullet))))\\
&\cong H^i((\operatorname{Hom}^\bullet(Y, \sigma_n(I^\bullet))))\\
&= 0
\end{aligned}
$$

for all $i > n$ and $Y \in \operatorname{Mod} A$.

§21. Universal coefficient theorems

Throughout this section, $\mathscr{A}$ is an abelian category, $\mathscr{I}$ (resp. $\mathscr{P}$) is the collection of injective (resp. projective) objects of $\mathscr{A}$ and $\mathscr{U}$ is the épaisse subcategory of $K(\mathscr{A})$ consisting of acyclic complexes. Also, R is a commutative ring and A is an R-algebra. We denote by E an injective cogenerator in Mod R and by D both $\mathrm{Hom}_R(-, E)$ and $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, E)$.

Lemma 21.1. *Let* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$. *Assume one of the following canonical exact sequences splits in* $\mathscr{A}^{\mathbb{Z}}$

$$0 \to B^\bullet(X^\bullet) \to X^\bullet \to Z'^\bullet(X^\bullet) \to 0,$$

$$0 \to Z^\bullet(X^\bullet) \to X^\bullet \to B'^\bullet(X^\bullet) \to 0.$$

Then $X^\bullet \cong H^\bullet(X^\bullet)$ *in* $D(\mathscr{A})$.

Proof. Assume the canonical exact sequence

$$0 \to Z^\bullet(X^\bullet) \xrightarrow{u} X^\bullet \xrightarrow{v} B'^\bullet(X^\bullet) \to 0$$

splits as an exact sequence in $\mathscr{A}^{\mathbb{Z}}$. Let $v' : B'^\bullet(X^\bullet) \to X^\bullet$ be a morphism in $\mathscr{A}^{\mathbb{Z}}$ with $v \circ v' = \mathrm{id}_{B'^\bullet(X^\bullet)}$. Let $w : B'^\bullet(X^\bullet) \to T(Z^\bullet(X^\bullet))$ be the inclusion. Then, since $d_X = Tu \circ w \circ v$, $d_X \circ v' = Tu \circ w$ and we get a morphism $\varphi = {}^t[-\,w \quad v'] : B'^\bullet(X^\bullet) \to C(u)$ in $C(\mathscr{A})$. Put $\varepsilon = [1 \quad 0] : C(u) \to T(Z^\bullet(X^\bullet))$ and $\hat{v} = [0 \quad v] : C(u) \to B'^\bullet(X^\bullet)$. Then $Q(\varphi) = Q(\hat{v})^{-1}$ and $-\,w = \varepsilon \circ \varphi$. Thus by Proposition 11.1(2) we have a triangle in $D(\mathscr{A})$

$$(Z^\bullet(X^\bullet),\ X^\bullet,\ B'^\bullet(X^\bullet), u, v, -\,w).$$

Then, since $T^{-1}(B'^\bullet(X^\bullet)) = B^\bullet(X^\bullet)$, by (TR2) we have a triangle in $D(\mathscr{A})$

$$(B^\bullet(X^\bullet),\ Z^\bullet(X^\bullet),\ X^\bullet, T^{-1}(w), u, v).$$

On the other hand, since $T^{-1}(w) : B^\bullet(X^\bullet) \to Z^\bullet(X^\bullet)$ is the inclusion, again by Proposition 11.1(2) we have a triangle in $D(\mathscr{A})$ of the form

$$(B^\bullet(X^\bullet),\ Z^\bullet(X^\bullet),\ H^\bullet(X^\bullet), T^{-1}(w), \cdot, \cdot).$$

It follows by Corollary 6.7 that $X^\bullet \cong H^\bullet(X^\bullet)$ in $D(\mathscr{A})$. In case the canonical exact sequence

$$0 \to B^\bullet(X^\bullet) \to X^\bullet \to Z'^\bullet(X^\bullet) \to 0$$

splits in $\mathscr{A}^{\mathbb{Z}}$, by the dual argument we conclude also that $X^\bullet \cong H^\bullet(X^\bullet)$ in $D(\mathscr{A})$.

Definition 21.1. For an abelian category $\mathscr{A}$ we set

$$\text{gl dim } \mathscr{A} = \sup\{n \geq 0 \mid \text{Ext}^n_{\mathscr{A}}(X, Y) \quad 0 \text{ for some } X, Y \in \text{Ob}(\mathscr{A})\},$$

which we call the global dimension of $\mathscr{A}$.

Lemma 21.2. *Assume* gl dim $\mathscr{A} \leq 1$. *Then the following hold.*
(1) $I^\bullet \cong H^\bullet(I^\bullet)$ *in* $D(\mathscr{A})$ *for all* $I^\bullet \in \text{Ob}(K(\mathscr{I}))$.
(2) $P^\bullet \cong H^\bullet(P^\bullet)$ *in* $D(\mathscr{A})$ *for all* $P^\bullet \in \text{Ob}(K(\mathscr{P}))$.

Proof. (1) Note first that $B^\bullet(I^\bullet) \in \text{Ob}(K(\mathscr{I}))$. Thus the canonical exact sequnce

$$0 \to B^\bullet(I^\bullet) \to I^\bullet \to Z'^\bullet(I^\bullet) \to 0$$

splits in $\mathscr{A}^{\mathbb{Z}}$ and Lemma 21.1 applies.
(2) Dual of (1).

Lemma 21.3. *Assume* gl dim $\mathscr{A} \leq 1$ *and* $\mathscr{A}$ *has either enough injectives or enough projectives*. *Then* $X^\bullet \cong H^\bullet(X^\bullet)$ *for all* $X^\bullet \in \text{Ob}(D(\mathscr{A}))$.

Proof. By Lemmas 16.2, 16.6 and 21.2.

Definition 21.2. For each $n \in \mathbb{Z}$ we set $H_n = H^{-n} : C(\mathscr{A}) \to \mathscr{A}$, called the n^{th} homology functor.

Proposition 21.4 (Universal coefficient theorem in cohomology). *Assume* $\mathscr{A}$ *has enough injectives. Then for any* $n \in \mathbb{Z}$ *there exists a natural exact sequence*

$$0 \to \text{Ext}^1_{\mathscr{A}}(H_{n-1}(X^\bullet), Y) \to \text{Ext}^n(X^\bullet, Y) \to \mathscr{A}(H_n(X^\bullet), Y) \to 0$$

for $X^\bullet \in \text{Ob}(D(\mathscr{A}))$ *and* $Y \in \text{Ob}(\mathscr{A})$ *with* inj dim $Y \leq 1$. *Furthermore, if* gl dim $\mathscr{A} \leq 1$, *then for any* $X^\bullet \in \text{Ob}(D(\mathscr{A}))$ *and* $n \in \mathbb{Z}$ *there exists a split exact sequence of functors on* $\mathscr{A}$

$$0 \to \text{Ext}^1_{\mathscr{A}}(H_{n-1}(X^\bullet), -) \to \text{Ext}^n(X^\bullet, -) \to \mathscr{A}(H_n(X^\bullet), -) \to 0.$$

Proof. Take an injective resolution $Y \to I^\bullet$ such that $I^i = 0$ for $i \geq 2$. Then $C(T^{-1} d_I^0) = I^\bullet$ and we have a triangle $(I^\bullet, I^0, I^1, \cdot, d_I^0, \cdot)$ in $K^+(\mathscr{A})$. Thus for any $n \in \mathbb{Z}$ we have an exact sequence

$$\mathrm{Ext}^{n-1}(X^\bullet, I^0) \to \mathrm{Ext}^{n-1}(X^\bullet, I^1) \to \mathrm{Ext}^n(X^\bullet, I^\bullet) \to \mathrm{Ext}^n(X^\bullet, I^0) \to \mathrm{Ext}^n(X^\bullet, I^1).$$

Also, by Proposition 18.8(1) we have

$$\begin{aligned}\mathrm{Ext}^n(X^\bullet, I^i) &\cong H^n(\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, I^i)) \\ &\cong H^n(\mathrm{Hom}^\bullet(X^\bullet, I^i)) \\ &\cong H^n(\mathscr{A}(X^\bullet, I^i)) \\ &\cong \mathscr{A}(H^{-n}(X^\bullet), I^i))\end{aligned}$$

for all $n \in \mathbb{Z}$ and $i = -1, 0$. Thus for any $n \in \mathbb{Z}$ we have

$$\mathrm{Cok}\,\mathrm{Ext}^{n-1}(X^\bullet, d_I^0) \cong \mathrm{Ext}^1_{\mathscr{A}}(H^{-n+1}(X^\bullet), Y),$$

$$\mathrm{Ker}\,\mathrm{Ext}^n(X^\bullet, d_I^0) \cong \mathscr{A}(H^{-n}(X^\bullet), Y).$$

Since $Y \cong I^\bullet$ in $D(\mathscr{A})$, we get a desired exact sequence. Next, assume gl dim $\mathscr{A} \leq 1$. Then by Lemma 21.3 $X^\bullet \cong H^\bullet(X^\bullet)$ in $D(\mathscr{A})$ and we have

$$\begin{aligned}\mathrm{Ext}^n(X^\bullet, X) &\cong \mathrm{Ext}^n(H^\bullet(X^\bullet), I^\bullet) \\ &\cong H^n(\boldsymbol{R}\,\mathrm{Hom}^\bullet(H^\bullet(X^\bullet), I^\bullet)) \\ &\cong H^n(\mathrm{Hom}^\bullet(H^\bullet(X^\bullet), I^\bullet)).\end{aligned}$$

Since

$$\mathrm{Hom}^n(H^\bullet(X^\bullet), I^\bullet) = \mathscr{A}(H^{-n}(X^\bullet), I^0) \oplus \mathscr{A}(H^{-n+1}(X^\bullet), I^1),$$

$$d^n_{\mathrm{Hom}^\bullet(H^\bullet(X^\bullet), I^\bullet)} = \begin{bmatrix} 0 & 0 \\ \mathscr{A}(H^{-n}(X^\bullet), d_I^0) & 0 \end{bmatrix},$$

we have

$$H^n(\mathrm{Hom}^\bullet(H^\bullet(X^\bullet), I^\bullet)) \cong \mathscr{A}(H^{-n}(X^\bullet), Y) \oplus \mathrm{Ext}^1_{\mathscr{A}}(H^{-n+1}(X^\bullet), Y)$$

and the required splitting follows.

Proposition 21.5 (Dual of Proposition 21.4). *Assume $\mathscr{A}$ has enough projectives. Then for* *any* $n \in \mathbb{Z}$ *there exists a natural exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathcal{A}}(X, H^{n-1}(Y^\bullet)) \to \operatorname{Ext}^n(X, Y^\bullet) \to \mathcal{A}(X, H^n(Y^\bullet)) \to 0$$

for $X \in \operatorname{Ob}(\mathcal{A})$ *with* proj dim $X \le 1$ *and* $Y^\bullet \in \operatorname{Ob}(D(\mathcal{A}))$. *Furthermore, if* gl dim $\mathcal{A} \le 1$, *then for any* $Y^\bullet \in \operatorname{Ob}(D(\mathcal{A}))$ *and* $n \in \mathbb{Z}$ *there exists a split exact sequence of functors on* $\mathcal{A}$

$$0 \to \operatorname{Ext}^1_{\mathcal{A}}(-, H^{n-1}(Y^\bullet)) \to \operatorname{Ext}^n(-, Y^\bullet) \to \mathcal{A}(-, H^n(Y^\bullet)) \to 0.$$

Lemma 21.6. (1) *Let* $I^\bullet, I'^\bullet \in \operatorname{Ob}(\mathcal{A}^{\mathbb{Z}})$ *be injective and* $u \in \mathcal{A}^{\mathbb{Z}}(I^\bullet, I'^\bullet)$. *Then, if* $X^\bullet \in \operatorname{Ob}(C(\mathcal{A}))$ *is acyclic, so is* $\operatorname{Hom}^\bullet(X^\bullet, C(T^{-1}u))$.

(2) *Let* $P^\bullet, P'^\bullet \in \operatorname{Ob}(\mathcal{A}^{\mathbb{Z}})$ *be projective and* $u \in \mathcal{A}^{\mathbb{Z}}(P'^\bullet, P^\bullet)$. *Then, if* $X^\bullet \in \operatorname{Ob}(C(\mathcal{A}))$ *is acyclic, so is* $\operatorname{Hom}^\bullet(C(u), X^\bullet)$.

Proof. (1) Let $n \in \mathbb{Z}$. Since by Lemma 18.3 we have

$$H^n(\operatorname{Hom}^\bullet(X^\bullet, C(T^{-1}u))) \cong K(\mathcal{A})(X^\bullet, T^n(C(T^{-1}u))),$$

it suffices to show $K(\mathcal{A})(X^\bullet, T^n(C(T^{-1}u))) = 0$. Since $T^n(I^\bullet), T^n(I'^\bullet) \in \operatorname{Ob}(\mathcal{A}^{\mathbb{Z}})$ are injective and $T^n(C(T^{-1}u)) \cong C(T^{-1}(T^n(u)))$, we may assume $n = 0$. Let ${}^t[f \quad g] \in C(\mathcal{A})(X^\bullet, C(T^{-1}u))$. We have $f \circ T^{-1}d_X = 0$ and $g \circ T^{-1}d_X = T^{-1}(u \circ f)$. Thus there exists $h \in \mathcal{A}^{\mathbb{Z}}(TX^\bullet, I^\bullet)$ such that $f = h \circ d_X$. Then $(g - T^{-1}(u \circ h)) \circ T^{-1}d_X = 0$ and there exists $h' \in \mathcal{A}^{\mathbb{Z}}(TX^\bullet, T^{-1}I'^\bullet)$ such that $g - T^{-1}(u \circ h) = h' \circ d_X$. It follows that

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} h \\ h' \end{bmatrix} d_X + \begin{bmatrix} 0 & 0 \\ T^{-1}u & 0 \end{bmatrix} \begin{bmatrix} T^{-1}h \\ T^{-1}h' \end{bmatrix}.$$

Thus ${}^t[f \quad g] \simeq 0$ and $K(\mathcal{A})(X^\bullet, T^n(C(T^{-1}u))) = 0$.

(2) Dual of (1).

Lemma 21.7. *Assume* gl dim $\mathcal{A} < \infty$. *Then the following hold*.

(1) *If* $I^\bullet \in \operatorname{Ob}(K(\mathcal{I}))$ *is acyclic, then* $I^\bullet = 0$ *in* $K(\mathcal{A})$ *and* $\operatorname{Hom}^\bullet(X^\bullet, I^\bullet)$ *is acyclic for all* $X^\bullet \in \operatorname{Ob}(K(\mathcal{A}))$.

(2) *If* $P^\bullet \in \operatorname{Ob}(K(\mathcal{P}))$ *is acyclic, then* $P^\bullet = 0$ *in* $K(\mathcal{A})$ *and* $\operatorname{Hom}^\bullet(P^\bullet, Y^\bullet)$ *is acyclic for all* $Y^\bullet \in \operatorname{Ob}(K(\mathcal{A}))$.

Proof. (1) Let $I^\bullet \in \operatorname{Ob}(\mathcal{U} \cap K(\mathcal{I}))$. Then we have $Z^\bullet(I^\bullet) \in \operatorname{Ob}(K(\mathcal{I}))$, $B^\bullet(I^\bullet) = Z^\bullet(I^\bullet)$ and $Z'^\bullet(I^\bullet) = T(Z^\bullet(I^\bullet))$. Thus we have an exact sequence in $C(\mathcal{A})$

$$0 \to Z^\bullet(I^\bullet) \xrightarrow{j} I^\bullet \xrightarrow{p} T(Z^\bullet(I^\bullet)) \to 0$$

with $d_I = Tj \circ p$. Since this exact sequence splits in $\mathscr{A}^{\mathbb{Z}}$, there exists $h \in \mathscr{A}^{\mathbb{Z}}(T(Z^\bullet(I^\bullet)), I^\bullet)$ such that $p \circ h = \mathrm{id}_{T(Z^\bullet(I^\bullet))}$. Then $[j \quad h] : C(\mathrm{id}_Z) \to I^\bullet$ is an isomorphism in $C(\mathscr{A})$ and by Proposition 3.5 $I^\bullet = 0$ in $K(\mathscr{A})$. It then follows by Lemma 18.3 that $\mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ is acyclic for all $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$.

(2) Dual of (1).

Proposition 21.8. *Assume* gl dim $\mathscr{A} < \infty$. *Then the following hold*.

(1) *If* $\mathscr{A}$ *has enough injectives and satisfies the condition Ab4**, *then* $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, I^\bullet) \cong \mathrm{Hom}^\bullet(X^\bullet, I^\bullet)$ *for all* $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$ *and* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$

(2) *If* $\mathscr{A}$ *has enough projectives and satisfies the condition Ab4, then* $\boldsymbol{R}\mathrm{Hom}^\bullet(P^\bullet, Y^\bullet) \cong \mathrm{Hom}^\bullet(P^\bullet, Y^\bullet)$ *for all* $P^\bullet \in \mathrm{Ob}(K(\mathscr{P}))$ *and* $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$

(3) *If* $\mathscr{A}$ *has both enough injectives and enough projectives, then the bi-∂-functor* $\mathrm{Hom}^\bullet : K(\mathscr{A})^{\mathrm{op}} \times K(\mathscr{A}) \to K(\mathrm{Mod}\ \mathbb{Z})$ *has a right derived functor*

$$\boldsymbol{R}\mathrm{Hom}^\bullet : D(\mathscr{A})^{\mathrm{op}} \times D(\mathscr{A}) \to D(\mathrm{Mod}\ \mathbb{Z})$$

such that $\boldsymbol{R}\mathrm{Hom}^\bullet(P^\bullet, I^\bullet) \cong \mathrm{Hom}^\bullet(P^\bullet, I^\bullet)$ *for all* $P^\bullet \in \mathrm{Ob}(K(\mathscr{P}))$ *and* $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$.

Proof. (1) Let $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$ and take a quasi-isomorphism $s : I^\bullet \to I'^\bullet$ with $I'^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$. Since $C(s)$ is acyclic, and since $C(s) \in \mathrm{Ob}(K(\mathscr{I}))$, it follows by Lemma 21.7(1) that $\mathrm{Hom}^\bullet(X^\bullet, s)$ is a quasi-isomorphism. Thus

$$\begin{aligned} \mathrm{Hom}^\bullet(X^\bullet, I^\bullet) &\cong \mathrm{Hom}^\bullet(X^\bullet, I'^\bullet) \\ &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, I'^\bullet) \\ &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, I^\bullet). \end{aligned}$$

(2) Dual of (1).

(3) The following Claims enable us to apply Proposition 17.3.

Claim 1: For any $X^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$, $K(\mathscr{I})$ satisfies the hypotheses of Propostion 13.6 for $\mathrm{Hom}^\bullet(X^\bullet, -) : K(\mathscr{A}) \to K(\mathrm{Mod}\ \mathbb{Z})$.

Proof. By Lemmas 16.2 and 21.7(1).

Claim 2: For any $Y^\bullet \in \mathrm{Ob}(K(\mathscr{A}))$, $K(\mathscr{P})$ satisfies the hypotheses of Propostion 13.6 for $\mathrm{Hom}^\bullet(-, Y^\bullet) : K(\mathscr{A}) \to K(\mathrm{Mod}\ \mathbb{Z})$.

Proof. By Lemmas 16.6 and 21.7(2).

Proposition 21.9. *Assume* gl dim $\mathscr{A} \le 1$. *Then the following hold.*

(1) *If* $\mathscr{A}$ *has enough injectives, then* $\boldsymbol{R}\mathrm{Hom}^{\bullet}(X^{\bullet}, I^{\bullet}) \cong \mathrm{Hom}^{\bullet}(X^{\bullet}, I^{\bullet})$ *for all* $X^{\bullet} \in \mathrm{Ob}(K(\mathscr{A}))$ *and* $I^{\bullet} \in \mathrm{Ob}(K(\mathscr{I}))$.

(2) *If* $\mathscr{A}$ *has enough projectives, then* $\boldsymbol{R}\mathrm{Hom}^{\bullet}(P^{\bullet}, Y^{\bullet}) \cong \mathrm{Hom}^{\bullet}(P^{\bullet}, Y^{\bullet})$ *for all* $P^{\bullet} \in \mathrm{Ob}(K(\mathscr{P}))$ *and* $Y^{\bullet} \in \mathrm{Ob}(K(\mathscr{A}))$.

Proof. (1) Let $X^{\bullet} \in \mathrm{Ob}(K(\mathscr{A}))$ and $I^{\bullet} \in \mathrm{Ob}(K(\mathscr{I}))$. By Lemmas 16.2 and 21.7(1) $K(\mathscr{I})$ satisfies the hypotheses of Propostion 13.6 for $\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : K(\mathscr{A}) \to K(\mathrm{Mod}\ \mathbb{Z})$ and we have $\boldsymbol{R}_{\mathrm{II}}\mathrm{Hom}^{\bullet}(X^{\bullet}, I^{\bullet}) \cong \mathrm{Hom}^{\bullet}(X^{\bullet}, I^{\bullet})$. Thus the next Claim completes the proof.

Claim : For any $Y^{\bullet} \in \mathrm{Ob}(D(\mathscr{A}))$, $\boldsymbol{R}_{\mathrm{II}}\mathrm{Hom}^{\bullet}(-, Y^{\bullet}) : K(\mathscr{A}) \to D(\mathrm{Mod}\ \mathbb{Z})$ vanishes on the acyclic complexes.

Proof. By Lemma 21.3 $Y^{\bullet} \cong H^{\bullet}(Y^{\bullet})$ in $D(\mathscr{A})$. Let

$$0 \to H^{\bullet}(Y^{\bullet}) \to I^{\bullet,0} \xrightarrow{d_2^{\bullet,0}} I^{\bullet,1} \to 0$$

be an exact sequence in $\mathscr{A}^{\mathbb{Z}}$ with $I^{\bullet,0}, I^{\bullet,1} \in \mathrm{Ob}(\mathscr{A}^{\mathbb{Z}})$ injective. Then, since by Proposition 15.12 $H^{\bullet}(Y^{\bullet}) \cong C(T^{-1}d_2^{\bullet,0})$ in $D(\mathscr{A})$, by Lemma 21.6(1) $\boldsymbol{R}_{\mathrm{II}}\mathrm{Hom}^{\bullet}(-, Y^{\bullet})$ vanishes on the acyclic complexes.

(2) Dual of (1).

Definition 21.3. A ring A is called left (rsp. right) hereditary if left gl dim $A \le 1$ (resp. right gl dim $A \le 1$).

Remark 21.1. For a ring A the following are equivalent.

(1) A is left hereditary.

(2) Every left ideal of A is projective.

Proposition 21.10 (Universal coefficient theorem in homology). *For any* $n \in \mathbb{Z}$ *there exists a natural exact sequence*

$$0 \to H_n(M^{\bullet}) \otimes_A X \to \mathrm{Tor}_n(M^{\bullet}, X) \to \mathrm{Tor}_1^A(H_{n-1}(M^{\bullet}), X) \to 0$$

for $M^{\bullet} \in \mathrm{Ob}(D(\mathrm{Mod}\ A^{\mathrm{op}}))$ *and* $X \in \mathrm{Mod}\ A$ *with* flat $\dim_A X \le 1$. *Furthermore, if A is right* *hereditary, then for any* $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $n \in \mathbb{Z}$ *there exists a split exact sequence of functors on* Mod A

$$0 \to H_n(M^\bullet) \otimes_A - \to \mathrm{Tor}_n(M^\bullet, -) \to \mathrm{Tor}_1^A(H_{n-1}(M^\bullet), -) \to 0.$$

Proof. Take a flat resolution $F^\bullet \to X$ such that $F^i = 0$ for $i \leq -2$. Then $C(d_F^{-1}) = F^\bullet$ and we have a triangle $(F^{-1}, F^0, F^\bullet, d_F^{-1}, \cdot, \cdot)$ in $K^-(\mathrm{Mod}\, A)$. Thus, since $M^\bullet \overset{L}{\otimes} -$ is a ∂-functor, for any $n \in \mathbb{Z}$ we have an exact sequence

$$\mathrm{Tor}_n(M^\bullet, F^{-1}) \to \mathrm{Tor}_n(M^\bullet, F^0) \to \mathrm{Tor}_n(M^\bullet, F^\bullet) \to \mathrm{Tor}_{n-1}(M^\bullet, F^{-1}) \to \mathrm{Tor}_{n-1}(M^\bullet, F^0).$$

Note also that

$$\begin{aligned}\mathrm{Tor}_n(M^\bullet, F^i) &\cong H^{-n}(M^\bullet \overset{L}{\otimes} F^i)\\ &\cong H^{-n}(M^\bullet \otimes F^i)\\ &\cong H^{-n}(M^\bullet \otimes_A F^i)\\ &\cong H^{-n}(M^\bullet) \otimes_A F^i\end{aligned}$$

for all $n \in \mathbb{Z}$ and $i = -1, 0$. Thus for any $n \in \mathbb{Z}$ we have

$$\mathrm{Cok}\,\mathrm{Tor}_n(M^\bullet, d_F^{-1}) \cong H^{-n}(M^\bullet) \otimes_A X,$$

$$\mathrm{Ker}\,\mathrm{Tor}_{n-1}(M^\bullet, d_F^{-1}) \cong \mathrm{Tor}_1^A(H^{-n+1}(M^\bullet), X).$$

Since $F^\bullet \cong X$ in $D(\mathrm{Mod}\, A)$, we get a desired exact sequence. Next, assume A is right hereditary. Then by Lemma 21.3 $M^\bullet \cong H^\bullet(M^\bullet)$ in $D(\mathrm{Mod}\, A^{\mathrm{op}})$ and we have

$$\begin{aligned}\mathrm{Tor}_n(M^\bullet, X) &\cong \mathrm{Tor}_n(H^\bullet(M^\bullet), F^\bullet)\\ &\cong H^{-n}(H^\bullet(M^\bullet) \overset{L}{\otimes} F^\bullet)\\ &\cong H^{-n}(H^\bullet(M^\bullet) \otimes F^\bullet).\end{aligned}$$

Since

$$[H^\bullet(M^\bullet) \otimes F^\bullet]^{-n} = (H^{-n+1}(M^\bullet) \otimes_A F^{-1}) \oplus (H^{-n}(M^\bullet) \otimes_A F^0),$$

$$d^{-n}_{H^\bullet(M^\bullet) \otimes F^\bullet} = (-1)^n \begin{bmatrix} 0 & 0 \\ H^{-n+1}(M^\bullet) \otimes d_F^{-1} & 0 \end{bmatrix},$$

we have

$$H^{-n}(H^\bullet(M^\bullet) \otimes F^\bullet) \cong \mathrm{Tor}_1^A(H^{-n+1}(M^\bullet), X) \oplus (H^{-n}(M^\bullet) \otimes_A X)$$

and the required splitting follows.

Lemma 21.11. (1) *Let* $P^\bullet, P'^\bullet \in \mathrm{Ob}((\mathrm{Mod}\, A^{\mathrm{op}})^{\mathbb{Z}})$ *be projective and* $u : P'^\bullet \to P^\bullet$ *a morphism in* $(\mathrm{Mod}\, A^{\mathrm{op}})^{\mathbb{Z}}$. *Then, if* $X^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A))$ *is acyclic, so is* $C(u) \otimes X^\bullet$.

(2) *Let* $P^\bullet, P'^\bullet \in \mathrm{Ob}((\mathrm{Mod}\, A)^{\mathbb{Z}})$ *be projective and* $u : P'^\bullet \to P^\bullet$ *a morphism in* $(\mathrm{Mod}\, A)^{\mathbb{Z}}$. *Then, if* $M^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A^{\mathrm{op}}))$ *is acyclic, so is* $M^\bullet \otimes C(u)$.

Proof. (1) By Lemma 19.5 $D(C(u) \otimes X^\bullet) \xrightarrow{\sim} \mathrm{Hom}^\bullet(C(u), D(X^\bullet))$ and Lemma 21.6(2) applies.

(2) By symmetry.

Lemma 21.12. (1) *If A has finite left global dimension and if* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A))$ *is acyclic, then* $P^\bullet = 0$ *in* $K(\mathrm{Mod}\, A)$ *and* $M^\bullet \otimes P^\bullet$ *is acyclic for all* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$.

(2) *If A has finite right global dimension and if* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}}))$ *is acyclic, then* $P^\bullet = 0$ *in* $K(\mathrm{Mod}\, A^{\mathrm{op}})$ *and* $P^\bullet \otimes X^\bullet$ *is acyclic for all* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$.

Proof. (1) By Lemmas 21.7(2) and 19.5(2).

(2) By symmetry.

Proposition 21.13. (1) *If A has finite left global dimension, then* $M^\bullet \overset{L}{\otimes} P^\bullet \cong M^\bullet \otimes P^\bullet$ *for all* $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A))$.

(2) *If A has finite right global dimension, then* $P^\bullet \overset{L}{\otimes} X^\bullet \cong P^\bullet \otimes X^\bullet$ *for all* $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$.

Proof. (1) Let $P^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A))$. By Proposition 12.20(1) we have a quasi-isomorphism $s : P'^\bullet \to P^\bullet$ with $P'^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$. Then, since $C(s)$ is acyclic, and since $C(s) \in \mathrm{Ob}(K(\mathrm{Proj}\, A))$, by Lemmas 19.3(1) and 21.12(1) $M^\bullet \otimes s$ is a quasi-isomorphism. Thus

$$\begin{aligned} M^\bullet \overset{L}{\otimes} P^\bullet &\cong M^\bullet \overset{L}{\otimes} P'^\bullet \\ &\cong M^\bullet \otimes P'^\bullet \\ &\cong M^\bullet \otimes P^\bullet. \end{aligned}$$

(2) By symmetry.

§22. Way-out functors

Throughout this section, $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are abelian categories. Also, R is a commutative ring and A is an R-algebra. We denote by E an injective cogenerator in Mod R and by D both $\mathrm{Hom}_R(-, E)$ and $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, E)$. Unless otherwise stated, functors are covariant functors.

Definition 22.1. Let $K^*(\mathscr{A})$ be a localizing subcategory of $D(\mathscr{A})$. A ∂-functor $F : D^*(\mathscr{A}) \to D(\mathscr{B})$ is called way-out right (resp. left) if for any $n_1 \in \mathbb{Z}$ there exists $n_2 \in \mathbb{Z}$ such that $H^i(F(X^\bullet)) = 0$ for $i < n_1$ (resp. $i > n_1$) and $X^\bullet \in \mathrm{Ob}(D^*(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2$ (resp. $i > n_2$), and is called way-out in both directions if both way-out left and way-out right.

In case F is contravariant, F is said to be way-out left, way-out right or way-out in both directions if so is the covariant ∂-functor $F : D^*(\mathscr{A})^{\mathrm{op}} \to D(\mathscr{B})$.

Proposition 22.1. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Assume there exists a subcollection $\mathscr{I}$ of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$,*

(2) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, $Y \in \mathscr{I}$ if and only if $Z \in \mathscr{I}$, and*

(3) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X, Y, Z \in \mathscr{I}$, then the induced sequence $0 \to FX \to FY \to FZ \to 0$ in $\mathscr{B}$ is exact.*

Then $\boldsymbol{R}^+F : D^+(\mathscr{A}) \to D(\mathscr{B})$, which exists by Corollary 13.7, is way-out right.

Proof. Let $n_1 \in \mathbb{Z}$ and put $n_2 = n_1 + 1$. Let $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2$. We claim that $\boldsymbol{R}^i F(X^\bullet) = 0$ for $i < n_1$. By Lemma 10.6 we have a quasi-isomorphism $X^\bullet \to \sigma_{\geq n_2}(X^\bullet)$ with $\sigma_{\geq n_2}(X^\bullet) \in \mathrm{Ob}(K^+(\mathscr{A}))$. Also, by Proposition 4.7 we have a quasi-isomorphism $\sigma_{\geq n_2}(X^\bullet) \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. By construction, we may assume $I^i = 0$ for $i < n_2 - 1 = n_1$. Thus for any $i < n_1$, since $F(I^\bullet)^i = F(I^i) = 0$, we have

$$\begin{aligned}\boldsymbol{R}^iF(X^\bullet) &\cong \boldsymbol{R}^iF(I^\bullet)\\ &\cong H^i(Q(F(I^\bullet)))\\ &= 0.\end{aligned}$$

Proposition 22.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Assume there exists a subcollection $\mathscr{I}$ of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any $X \in \mathrm{Ob}(\mathscr{A})$ there exists a monomorphism $X \to I$ in $\mathscr{A}$ with $I \in \mathscr{I}$,*

(2) *if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $\mathscr{A}$ with $X \in \mathscr{I}$, $Y \in \mathscr{I}$ if and only if $Z \in \mathscr{I}$,*

(3) *there exists an integer $n \geq 1$ such that if*

$$X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0$$

is an exact sequence in $\mathscr{A}$ *with* $X^0, X^1, \cdots, X^{n-1} \in \mathscr{I}$ *then* $X^n \in \mathscr{I}$, *and*

(4) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{I}$, *then the induced sequence* $0 \to FX \to FY \to FZ \to 0$ *in* $\mathscr{B}$ *is exact*.

Then $\boldsymbol{R}F : D(\mathscr{A}) \to D(\mathscr{B})$, *which exists by Proposition 16.4, is way-out in both directions.*

Proof. Note that by Proposition 16.4 $\boldsymbol{R}F|_{D^+(\mathscr{A})} \cong \boldsymbol{R}^+F$. Thus by Proposition 22.1 $\boldsymbol{R}F$ is way-out right. Next, let $n_1 \in \mathbb{Z}$ and put $n_2 = n_1 - n$. Let $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i > n_2$. We claim that $\boldsymbol{R}^iF(X^\bullet) = 0$ for $i > n_1$. By Lemma 16.2 we have a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I}))$. Since $H^i(I^\bullet) = 0$ for $i \geq n_1 > n_2$, by Lemma 10.7 we have a quasi-isomorphism $\sigma_{\leq n_1}(I^\bullet) \to I^\bullet$. Also, since we have an exact sequence

$$I^{n_2} \to \cdots \to I^{n_1-1} \to Z^{n_1}(I^\bullet) \to 0$$

with $n_1 - n_2 = n$, $Z^{n_1}(I^\bullet) \in \mathrm{Ob}(K(\mathscr{I}))$ and $\sigma_{\leq n_1}(I^\bullet) \in \mathrm{Ob}(K(\mathscr{I}))$. Thus for any $i > n_1$, since $F(\sigma_{\leq n_1}(I^\bullet))^i = 0$, we have

$$\begin{aligned} \boldsymbol{R}^iF(X^\bullet) &\cong \boldsymbol{R}^iF(\sigma_{\leq n_1}(I^\bullet)) \\ &\cong H^i(Q(F(\sigma_{\leq n_1}(I^\bullet)))) \\ &= 0. \end{aligned}$$

Proposition 22.3. *Let* $F : \mathscr{A} \to \mathscr{B}$ *be a left exact functor. Assume there exists a subcollection* $\mathscr{I}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists a monomorphism* $X \to I$ *in* $\mathscr{A}$ *with* $I \in \mathscr{I}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X \in \mathscr{I}$, $Y \in \mathscr{I}$ *if and only if* $Z \in \mathscr{I}$,

(3) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $X, Y, Z \in \mathscr{I}$, *then the induced sequence* $0 \to FX \to FY \to FZ \to 0$ *in* $\mathscr{B}$ *is exact, and*

(4) *F has finite cohomological dimension on* $\mathscr{A}$, *i.e., there exists* $n \geq 1$ *such that* $\boldsymbol{R}^iF$ *vanishes on* $\mathscr{A}$ *for* $i > n$ *(Note that by Corollary 13.7* $\boldsymbol{R}^+F$ *exists).*

Then $\boldsymbol{R}F : D(\mathscr{A}) \to D(\mathscr{B})$, *which exists by Corollary 16.5, is way-out in both directions.*

Proof. By Corollary 16.5 and Proposition 22.2.

Proposition 22.4 (Dual of Proposition 22.1). *Let* $G : \mathscr{A} \to \mathscr{B}$ *be an additive functor. Assume there exists a subcollection* $\mathscr{P}$ *of* $\mathrm{Ob}(\mathscr{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathscr{A})$ *there exists an epimorphism* $P \to X$ *in* $\mathscr{A}$ *with* $P \in \mathscr{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathscr{A}$ *with* $Z \in \mathscr{P}$, *then* $Y \in \mathscr{P}$ *if and only*

if $X \in \mathcal{P}$, *and*

(3) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $X, Y, Z \in \mathcal{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact.*

Then $\boldsymbol{L}^{-}G : D^{-}(\mathcal{A}) \to D(\mathcal{B})$, *which exists by Corollary 14.7, is way-out left.*

Proposition 22.5 (Dual of Proposition 22.2). *Let* $G : \mathcal{A} \to \mathcal{B}$ *be a right exact functor. Assume there exists a subcollection* $\mathcal{P}$ *of* $\mathrm{Ob}(\mathcal{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathcal{A})$ *there exists a monomorphism* $P \to X$ *in* $\mathcal{A}$ *with* $P \in \mathcal{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $Z \in \mathcal{P}$, *then* $Y \in \mathcal{P}$ *if and only if* $X \in \mathcal{P}$,

(3) *there exists an integer* $n \geq 1$ *such that if*

$$0 \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^{0}$$

is an exact sequence in $\mathcal{A}$ *with* $X^{0}, X^{-1}, \cdots, X^{-n+1} \in \mathcal{P}$ *then* $X^{-n} \in \mathcal{P}$, *and*

(4) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $X, Y, Z \in \mathcal{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact.*

Then $\boldsymbol{L}G : D(\mathcal{A}) \to D(\mathcal{B})$, *which exists by Proposition 16.8, is way-out in both directions.*

Proposition 22.6 (Dual of Proposition 22.3). *Let* $G : \mathcal{A} \to \mathcal{B}$ *be a right exact functor. Assume there exists a subcollection* $\mathcal{P}$ *of* $\mathrm{Ob}(\mathcal{A})$ *such that*

(1) *for any* $X \in \mathrm{Ob}(\mathcal{A})$ *there exists a monomorphism* $P \to X$ *in* $\mathcal{A}$ *with* $P \in \mathcal{P}$,

(2) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $Z \in \mathcal{P}$, *then* $Y \in \mathcal{P}$ *if and only if* $X \in \mathcal{P}$,

(3) *if* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $X, Y, Z \in \mathcal{P}$, *then the induced sequence* $0 \to GX \to GY \to GZ \to 0$ *is exact, and*

(4) *G has finite homological dimension on* $\mathcal{A}$, *i.e., there exists* $n \geq 1$ *such that* $\boldsymbol{L}_iG$ *vanishes on* $\mathcal{A}$ *for all* $i > n$ (*Note that by Corollary 14.7* $\boldsymbol{L}^{-}G$ *exists*).

Then $\boldsymbol{L}G : D(\mathcal{A}) \to D(\mathcal{B})$, *which exists by Corollary 16.9, is way-out in both directions.*

Throughout the rest of this section, $\mathcal{I}$ (resp. $\mathcal{P}$) is the collection of injective (resp. projective) objects of $\mathcal{A}$. We denote by $K(\mathcal{I})_{\mathrm{L}}$ (resp. $K(\mathcal{P})_{\mathrm{L}}$) the full subcategory of $K(\mathcal{I})$ (resp. $K(\mathcal{P})$) consisting of $\mathcal{U}$-local (resp. $\mathcal{U}$-colocal) complexes, where $\mathcal{U}$ is the épaisse subcategory of $K(\mathcal{A})$ consisting of acyclic complexes.

Proposition 22.7. *Assume* $\mathcal{A}$ *has enough injectives. Then for* $X^{\bullet} \in \mathrm{Ob}(D(\mathcal{A}))$ *the following hold.*

(1) *If* $X^{\bullet} \in \mathrm{Ob}(D^{+}(\mathcal{A}))$, *then* $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(-, X^{\bullet}) : D(\mathcal{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out right.*

(2) *If* $\mathcal{A}$ *satisfies the condition Ab4** *and if* $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(-, X^{\bullet}) : D(\mathcal{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out*

right, then $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$.

Proof. (1) Let $n_1 \in \mathbb{Z}$. Take $n \in \mathbb{Z}$ such that $H^i(X^\bullet) = 0$ for $i < n$ and put $n_2 = (n-1) - n_1$. By Lemma 10.6 we have a quasi-isomorphism $X^\bullet \to \sigma_{\geq n}(X^\bullet)$. Also, by Proposition 4.7 we have a quasi-isomorphism $\sigma_{\geq n}(X^\bullet) \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ such that $I^i = 0$ for $i < n-1$. Let $Y^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(Y^\bullet) = 0$ for $i > n_2$. Since $H^i(Y^\bullet) = 0$ for $i > n_2$, by Lemma 10.7 we have a quasi-isomorphism $\sigma_{\leq n_2}(Y^\bullet) \to Y^\bullet$. Thus for $i < n_1$, since $T^i(I^\bullet)^j = I^{i+j} = 0$ for $j \leq n_2$ and $\sigma_{\leq n_2}(Y^\bullet)^j = 0$ for $j > n_2$, by Proposition 10.12 we have

$$\begin{aligned}\mathrm{Ext}^i(Y^\bullet,\ X^\bullet) &\cong \mathrm{Ext}^i(\sigma_{\leq n_2}(Y^\bullet),\ I^\bullet)\\ &\cong D(\mathscr{A})(\sigma_{\leq n_2}(Y^\bullet), T^i(I^\bullet)))\\ &\cong K(\mathscr{A})(\sigma_{\leq n_2}(Y^\bullet), T^i(I^\bullet)))\\ &= 0.\end{aligned}$$

(2) By Proposition 12.15(2) we have a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K(\mathscr{I})_{\mathrm{L}})$. For $n_1 = 0$, there exists $n_2 = n \in \mathbb{Z}$ such that $\mathrm{Ext}^i(Y^\bullet,\ X^\bullet) = 0$ for $i < 0$ and $Y^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(Y^\bullet) = 0$ for $i > n$. Let $i < n$ and $j : Z^i(I^\bullet) \to I^i$ the inclusion. Then, since by Proposition 9.13(2) we have

$$\begin{aligned}K(\mathscr{A})(T^{-i}(Z^i(I^\bullet)),\ I^\bullet) &\cong D(\mathscr{A})(T^{-i}(Z^i(I^\bullet)),\ I^\bullet)\\ &\cong D(\mathscr{A})(T^{-i}(Z^i(I^\bullet)),\ X^\bullet)\\ &\cong D(\mathscr{A})(T^{-n}(Z^i(I^\bullet)), T^{i-n}(X^\bullet))\\ &\cong \mathrm{Ext}^{i-n}(T^{-n}(Z^i(I^\bullet)),\ X^\bullet)\\ &= 0,\end{aligned}$$

there exists $f : Z^i(I^\bullet) \to I^{i-1}$ such that $j = d_I^{i-1} \circ f$. It follows that $B^i(I^\bullet) = Z^i(I^\bullet)$. Consequently, $H^i(X^\bullet) \cong H^i(I^\bullet) = 0$ for all $i < n$ and $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$.

Proposition 22.8. *Assume $\mathscr{A}$ has enough injectives. Then for $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A}))$ the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(D^+(\mathscr{A})_{\mathrm{fid}})$.

(2) *There exists a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$.*

(3) $\boldsymbol{R}\mathrm{Hom}^\bullet(-,\ X^\bullet) : D(\mathscr{A}) \to D(\mathrm{Mod}\ \mathbb{Z})$ *is way-out left (and thus by Proposition 22.7(1) way-out in both directions).*

Proof. (1) $\Rightarrow$ (2). By Proposition 11.12.

(2) $\Rightarrow$ (3). Take $n \in \mathbb{Z}$ such that $I^i = 0$ for $i > n$. Let $n_1 \in \mathbb{Z}$ and put $n_2 = n - n_1 + 1$. Let $Y^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(Y^\bullet) = 0$ for $i < n_2$. By Lemma 10.6 we have a quasi-isomorphism $Y^\bullet \to \sigma_{\geq n_2}(Y^\bullet)$. Thus for any $i > n_1$, since $T^i(I^\bullet)^j = I^{i+j} = 0$ for $j \geq n_2 - 1$ and $\sigma_{\geq n_2}(Y^\bullet)^j = 0$

for $j < n_2 - 1$, by Proposition 10.12 we have

$$\begin{aligned}\mathrm{Ext}^i(Y^\bullet, X^\bullet) &\cong \mathrm{Ext}^i(\sigma_{\geq n_2}(Y^\bullet), I^\bullet)\\ &\cong D(\mathcal{A})(\sigma_{\geq n_2}(Y^\bullet), T^i(I^\bullet))\\ &\cong K(\mathcal{A})(\sigma_{\geq n_2}(Y^\bullet), T^i(I^\bullet))\\ &= 0.\end{aligned}$$

(3) $\Rightarrow$ (1). For $n_1 = 0$, there exists $n_2 = n \in \mathbb{Z}$ such that $\mathrm{Ext}^i(Y^\bullet, X^\bullet) = 0$ for $i > 0$ and $Y^\bullet \in \mathrm{Ob}(D(\mathcal{A}))$ with $H^i(Y^\bullet) = 0$ for $i < n$. Thus for any $i > n$ and $Y \in \mathrm{Ob}(\mathcal{A})$ we have

$$\begin{aligned}\mathrm{Ext}^i(Y, X^\bullet) &\cong \mathrm{Ext}^{i-n}(T^{-n}Y, X^\bullet)\\ &= 0.\end{aligned}$$

Proposition 22.9. *Assume* $\mathcal{A}$ *has enough injectives and satisfies the condition Ab4**. *Then for* $X^\bullet \in \mathrm{Ob}(D(\mathcal{A}))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(D(\mathcal{A})_{\mathrm{fid}})$.

(2) *There exists a quasi-isomorphism* $X^\bullet \to I^\bullet$ *with* $I^\bullet \in \mathrm{Ob}(K^-(\mathcal{I})_{\mathrm{L}})$.

(3) $\boldsymbol{R}\mathrm{Hom}^\bullet(-, X^\bullet) : D(\mathcal{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out left.*

Proof. (1) $\Rightarrow$ (2). By Proposition 12.15(2).

(2) $\Rightarrow$ (3). Take $n \in \mathbb{Z}$ such that $I^i = 0$ for $i > n$. Let $n_1 \in \mathbb{Z}$ and put $n_2 = n - n_1 + 1$. Let $Y^\bullet \in \mathrm{Ob}(D(\mathcal{A}))$ with $H^i(Y^\bullet) = 0$ for $i < n_2$. By Lemma 10.6 we have a quasi-isomorphism $Y^\bullet \to \sigma_{\geq n_2}(Y^\bullet)$. Thus for any $i > n_1$, since $T^i(I^\bullet)^j = I^{i+j} = 0$ for $j \geq n_2 - 1$ and $\sigma_{\geq n_2}(Y^\bullet)^j = 0$ for $j < n_2 - 1$, by Proposition 9.13(2) we have

$$\begin{aligned}\mathrm{Ext}^i(Y^\bullet, X^\bullet) &\cong \mathrm{Ext}^i(\sigma_{\geq n_2}(Y^\bullet), I^\bullet)\\ &\cong D(\mathcal{A})(\sigma_{\geq n_2}(Y^\bullet), T^i(I^\bullet))\\ &\cong K(\mathcal{A})(\sigma_{\geq n_2}(Y^\bullet), T^i(I^\bullet))\\ &= 0.\end{aligned}$$

(3) $\Rightarrow$ (1). Same as in the proof of Proposition 22.8.

Remark 22.1. (1) Assume $\mathcal{A}$ has enough injectives. Then it follows by Proposition 22.8 that $D^+(\mathcal{A})_{\mathrm{fid}} \subset D^{\mathrm{b}}(\mathcal{A})$.

(2) Assume $\mathcal{A}$ has enough injectives and satisfies the condition Ab4*. Then it follows by Proposition 22.9 that $D(\mathcal{A})_{\mathrm{fid}} \subset D^-(\mathcal{A})$.

Proposition 22.10. *Assume* $\mathcal{A}$ *has enough injectives and satisfies the condition Ab4**. *Then for* $X^\bullet \in \mathrm{Ob}(D(\mathcal{A}))$ *the following are equivalent.*

(1) $X^{\bullet} \in \mathrm{Ob}(D^{+}(\mathscr{A})_{\mathrm{fid}})$.

(2) *There exists a quasi-isomorphism* $X^{\bullet} \to I^{\bullet}$ *with* $I^{\bullet} \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$.

(3) $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(-, X^{\bullet}) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out in both directions.*

Proof. (1) ⇒ (2) ⇒ (3). By Proposition 22.8.

(3) ⇒ (1). By Propositions 22.7(2) and 22.9.

Proposition 22.11 (Dual of Proposition 22.7). *Assume* $\mathscr{A}$ *has enough projectives. Then for* $X^{\bullet} \in \mathrm{Ob}(D(\mathscr{A}))$ *the following hold.*

(1) *If* $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathscr{A}))$*, then* $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out right.*

(2) *If* $\mathscr{A}$ *satisfies the condition Ab4 and if* $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out right, then* $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathscr{A}))$.

Proposition 22.12 (Dual of Proposition 22.8). *Assume* $\mathscr{A}$ *has enough projectives. Then for* $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathscr{A}))$ *the following are equivalent.*

(1) $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathscr{A})_{\mathrm{fpd}})$.

(2) *There exists a quasi-isomorphism* $P^{\bullet} \to X^{\bullet}$ *with* $P^{\bullet} \in \mathrm{Ob}(D^{\mathrm{b}}(\mathscr{P}))$.

(3) $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out left (and thus by Proposition 22.11(1) way-out in both directions).*

Proposition 22.13 (Dual of Proposition 22.9). *Assume* $\mathscr{A}$ *has enough projectives and satisfies the condition Ab4. Then for* $X^{\bullet} \in \mathrm{Ob}(D(\mathscr{A}))$ *the following are equivalent.*

(1) $X^{\bullet} \in \mathrm{Ob}(D(\mathscr{A})_{\mathrm{fpd}})$.

(2) *There exists a quasi-isomorphism* $P^{\bullet} \to X^{\bullet}$ *with* $P^{\bullet} \in \mathrm{Ob}(D^{+}(\mathscr{P})_{\mathrm{L}})$.

(3) $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out left.*

Remark 22.2. (1) Assume $\mathscr{A}$ has enough projectives. Then it follows by Proposition 22.12 that $D^{-}(\mathscr{A})_{\mathrm{fpd}} \subset D^{\mathrm{b}}(\mathscr{A})$.

(2) Assume $\mathscr{A}$ has enough projectives and satisfies the condition Ab4. Then it follows by Proposition 22.13 that $D(\mathscr{A})_{\mathrm{fpd}} \subset D^{+}(\mathscr{A})$.

Proposition 22.14 (Dual of Proposition 22.10). *Assume* $\mathscr{A}$ *has enough projectives and satisfies the condition Ab4. Then for* $X^{\bullet} \in \mathrm{Ob}(D(\mathscr{A}))$ *the following are equivalent.*

(1) $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathscr{A})_{\mathrm{fpd}})$.

(2) *There exists a quasi-isomorphism* $P^{\bullet} \to X^{\bullet}$ *with* $P^{\bullet} \in \mathrm{Ob}(D^{\mathrm{b}}(\mathscr{P}))$.

(3) $\boldsymbol{R}\,\mathrm{Hom}^{\bullet}(X^{\bullet}, -) : D(\mathscr{A}) \to D(\mathrm{Mod}\,\mathbb{Z})$ *is way-out in both directions.*

Proposition 22.15. *For* $X^{\bullet} \in \mathrm{Ob}(D(\mathrm{Mod}\,A))$ *the following are equivalent.*

(1) $X^{\bullet} \in \mathrm{Ob}(D^{-}(\mathrm{Mod}\,A))$.

(2) *There exists a quasi-isomorphism* $P^\bullet \to X^\bullet$ *with* $P^\bullet \in \mathrm{Ob}(K^-(\mathrm{Flat}\,A))$.

(3) $- \overset{L}{\otimes} X^\bullet : D(\mathrm{Mod}\,A^{\mathrm{op}}) \to D(\mathrm{Mod}\,R)$ *is way-out left*.

Proof. (1) ⇒ (2). Take $n \in \mathbb{Z}$ such that $H^i(X^\bullet) = 0$ for $i > n$. Then by Lemma 10.7 the canonical monomorphism $\sigma_{n}(X^\bullet) \to X^\bullet$ is a quasi-isomorphism. Also, by Proposition 4.11 we have a quasi-isomorphism $P^\bullet \to \sigma_{n}(X^\bullet)$ with $P^\bullet \in \mathrm{Ob}(K^-(\mathrm{Proj}\,A))$.

(1) ⇒ (2). Let $n_1 \in \mathbb{Z}$. Take $n \in \mathbb{Z}$ such that $P^i = 0$ for $i > n$ and put $n_2 = n_1 - n$. Let $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A^{\mathrm{op}}))$ with $H^i(M^\bullet) = 0$ for $i > n_2$. By Lemma 10.7 we have a quasi-isomorphism $\sigma_{\leq n_2}(M^\bullet) \to M^\bullet$. Thus for $i < n_1$, since $[\sigma_{\leq n_2}(M^\bullet) \otimes P^\bullet]^i = 0$, we have

$$
\begin{aligned}
H^i(M^\bullet \overset{L}{\otimes} X^\bullet) &\cong H^i(\sigma_{\leq n_2}(M^\bullet) \overset{L}{\otimes} P^\bullet) \\
&\cong H^i(\sigma_{\leq n_2}(M^\bullet) \otimes P^\bullet) \\
&= 0.
\end{aligned}
$$

(3) ⇒ (1). For $n_1 = 0$, there exists $n_2 = n \in \mathbb{Z}$ such that $H^i(M^\bullet \overset{L}{\otimes} X^\bullet) = 0$ for $i > 0$ and $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A^{\mathrm{op}}))$ with $H^i(M^\bullet) = 0$ for $i > n$. Thus for any $i > -n$ we have

$$
\begin{aligned}
H^i(X^\bullet) &\cong H^i(A \overset{L}{\otimes} X^\bullet) \\
&\cong H^{i+n}(T^{-n}(A) \overset{L}{\otimes} X^\bullet) \\
&= 0.
\end{aligned}
$$

Proposition 22.16. *For* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A)_{\mathrm{fTd}})$.

(2) *There exists an isomorphism* $P^\bullet \to X^\bullet$ *in* $D(\mathrm{Mod}\,A)$ *with* $P^\bullet \in \mathrm{Ob}(K^+(\mathrm{Flat}\,A))$.

(3) $- \overset{L}{\otimes} X^\bullet : D(\mathrm{Mod}\,A^{\mathrm{op}}) \to D(\mathrm{Mod}\,R)$ *is way-out right*.

Proof. (1) ⇒ (2). By Lemma 20.7.

(2) ⇒ (3). Let $n_1 \in \mathbb{Z}$. Take $n \in \mathbb{Z}$ such that $P^i = 0$ for $i < n$ and put $n_2 = n_1 - n + 1$. Let $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A^{\mathrm{op}}))$ with $H^i(M^\bullet) = 0$ for $i < n_2$. By Lemma 10.6 we have a quasi-isomorphism $M^\bullet \to \sigma_{\geq n_2}(M^\bullet)$. Thus for $i < n_1$, since $[\sigma_{\leq n_2}(M^\bullet) \otimes P^\bullet]^i = 0$, we have

$$
\begin{aligned}
H^i(M^\bullet \overset{L}{\otimes} X^\bullet) &\cong H^i(\sigma_{\leq n_2}(M^\bullet) \overset{L}{\otimes} P^\bullet) \\
&\cong H^i(\sigma_{\leq n_2}(M^\bullet) \otimes P^\bullet) \\
&= 0.
\end{aligned}
$$

(3) ⇒ (1). For $n_1 = 0$, there exists $n_2 = n \in \mathbb{Z}$ such that $H^i(M^\bullet \overset{L}{\otimes} X^\bullet) = 0$ for $i < 0$ and $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\,A^{\mathrm{op}}))$ with $H^i(M^\bullet) = 0$ for $i < n$. Thus for any $i > n$ and $M \in \mathrm{Mod}\,A^{\mathrm{op}}$ we have

$$
\begin{aligned}
\operatorname{Tor}_i(M,\ X^\bullet) &\cong H^{-i}(M \overset{L}{\otimes} X^\bullet) \\
&\cong H^{n-i}(T^{-n}(M) \overset{L}{\otimes} X^\bullet) \\
&= 0.
\end{aligned}
$$

Proposition 22.17. *For* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A))$ *the following are equivalent.*

(1) $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ A)_{\mathrm{fTd}})$.

(2) *There exists an isomorphism* $P^\bullet \to X^\bullet$ *in* $D(\mathrm{Mod}\ A)$ *with* $P^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Flat}\ A))$.

(3) $- \overset{L}{\otimes} X^\bullet : D(\mathrm{Mod}\ A^{\mathrm{op}}) \to D(\mathrm{Mod}\ R)$ *is way-out in both directions*.

Proof. By Propositions 22.15 and 22.16.

Remark 22.3. (1) It follows by Proposition 22.17 that $D^-(\mathrm{Mod}\ A)_{\mathrm{fTd}} \subset D^{\mathrm{b}}(\mathrm{Mod}\ A)$.

(2) It follows by Propositions 22.13 and 22.17 that $D^-(\mathrm{Mod}\ A)_{\mathrm{fpd}} \subset D^-(\mathrm{Mod}\ A)_{\mathrm{fTd}}$.

(3) It follows by Proposition 22.16 that $D(\mathrm{Mod}\ A)_{\mathrm{fTd}} \subset D^+(\mathrm{Mod}\ A)$.

(4) It follows by Propositions 22.12 and 22.16 that $D(\mathrm{Mod}\ A)_{\mathrm{fpd}} \subset D(\mathrm{Mod}\ A)_{\mathrm{fTd}}$.

§23. Lemma on way-out functors

Throughout this section, $\mathscr{A}$, $\mathscr{B}$ are abelian categories and $\mathscr{A}'$, $\mathscr{B}'$ are thick subcategories of $\mathscr{A}$ and $\mathscr{B}$, respectively. Unless stated otherwise, functors are covariant.

Lemma 23.1. *For any $X^\bullet \in C(\mathscr{A})$ and $n \in \mathbb{Z}$, there exists a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \sigma'_{<n}(X^\bullet) & \to & \sigma_{\le n}(X^\bullet) & \to & T^{-n}(H^n(X^\bullet)) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \sigma'_{<n}(X^\bullet) & \to & X^\bullet & \to & \sigma'_{\ge n}(X^\bullet) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \sigma_{>n}(X^\bullet) & = & \sigma_{>n}(X^\bullet) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,. & &
\end{array}$$

Proof. Straightforward.

Lemma 23.2. *For any $X^\bullet \in C(\mathscr{A})$ and $n \in \mathbb{Z}$, there exists a commutative diagram with exact rows and column*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \sigma_{<n}(X^\bullet) & \to & \sigma'_{<n}(X^\bullet) & \to & C(\mathrm{id}_{T^{-n}(B^n(X^\bullet))}) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \sigma_{<n}(X^\bullet) & \to & X^\bullet & \to & \sigma_{\ge n}(X^\bullet) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \sigma'_{\ge n}(X^\bullet) & = & \sigma'_{\ge n}(X^\bullet) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,. & &
\end{array}$$

Proof. Straightforward.

Definition 23.1. For each $n \in \mathbb{Z}$, we define truncation functors $\tau_{>n}, \tau_{\leq n} : C(\mathscr{A}) \to C(\mathscr{A})$ as follows:

$$\tau_{>n}(X^\bullet)^i = \begin{cases} X^i & (i > n) \\ 0 & (i \leq n) \end{cases}, \quad \tau_{\leq n}(X^\bullet)^i = \begin{cases} 0 & (i > n) \\ X^i & (i \leq n) \end{cases}$$

for $X^\bullet \in C(\mathscr{A})$. We set $\tau_{\geq n} = \tau_{>n-1}$ and $\tau_{<n} = \tau_{\leq n-1}$.

Lemma 23.3. *For any $X^\bullet \in C(\mathscr{A})$ and $n \in \mathbb{Z}$, there exists a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \tau_{>n}(X^\bullet) & \to & \tau_{\geq n}(X^\bullet) & \to & T^{-n}(X^n) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \tau_{>n}(X^\bullet) & \to & X^\bullet & \to & \tau_{\leq n}(X^\bullet) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \tau_{<n}(X^\bullet) & = & \tau_{<n}(X^\bullet) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Proof. Straightforward.

Lemma 23.4. *For any $X^\bullet \in C(\mathscr{A})$ and $n \in \mathbb{Z}$, there exist triangles in $D(\mathscr{A})$ of the following form*

(1) $(\tau_{>n}(X^\bullet), X^\bullet, \tau_{\geq n}(X^\bullet), \cdot, \cdot, \cdot)$,

(2) $(\tau_{>n}(X^\bullet), \tau_{\geq n}(X^\bullet), T^{-n}(X^n), \cdot, \cdot, \cdot)$,

(3) $(T^{-n}(X^n), \tau_{\leq n}(X^\bullet), \tau_{<n}(X^\bullet), \cdot, \cdot, \cdot)$,

(4) $(\sigma_{\leq n}(X^\bullet), X^\bullet, \sigma_{>n}(X^\bullet), \cdot, \cdot, \cdot)$,

(5) $(\sigma_{<n}(X^\bullet), \sigma_{\leq n}(X^\bullet), T^{-n}(H^n(X^\bullet)), \cdot, \cdot, \cdot)$,

(6) $(T^{-n}(H^n(X^\bullet)), \sigma_{\geq n}(X^\bullet), \sigma_{>n}(X^\bullet), \cdot, \cdot, \cdot)$.

Proof. By Lemma 23.3 we have exact sequences in $C(\mathscr{A})$

$$0 \to \tau_{>n}(X^\bullet) \to X^\bullet \to \tau_{\geq n}(X^\bullet) \to 0,$$
$$0 \to \tau_{>n}(X^\bullet) \to \tau_{\geq n}(X^\bullet) \to T^{-n}(X^n) \to 0,$$
$$0 \to T^{-n}(X^n) \to \tau_{\leq n}(X^\bullet) \to \tau_{<n}(X^\bullet) \to 0.$$

Thus by Proposition 11.1(2) we get first three triangles. Also, by Lemma 23.1 we have exact sequences in $C(\mathscr{A})$

$$0 \to \sigma_{\leq n}(X^\bullet) \to X^\bullet \to \sigma_{>n}(X^\bullet) \to 0,$$
$$0 \to \sigma'_{<n}(X^\bullet) \to \sigma_{\leq n}(X^\bullet) \to T^{-n}(H^n(X^\bullet)) \to 0,$$
$$0 \to T^{-n}(H^n(X^\bullet)) \to \sigma'_{\geq n}(X^\bullet) \to \sigma_{>n}(X^\bullet) \to 0.$$

Thus, since by Lemma 23.2 we have isomorphisms in $D(\mathscr{A})$

$$\sigma_{<n}(X^\bullet) \xrightarrow{\sim} \sigma'_{<n}(X^\bullet), \quad \sigma_{\geq n}(X^\bullet)) \xrightarrow{\sim} \sigma'_{\geq n}(X^\bullet),$$

by Proposition 11.1(2) we get last three triangles.

Proposition 23.5 (Lemma on way-out functors). *Let F, G : $D^*_{\mathscr{A}'}(\mathscr{A}) \to D(\mathscr{B})$ be ∂-functors, where* $* = +, -,$ b *or nothing, and* $\eta \in$ Hom (F, G). *Then the following hold.*

(1) *Assume $\eta(X)$ is an isomorphism for all $X \in$* Ob($\mathscr{A}$'). *Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A})$).

(2) *Assume $\eta(X)$ is an isomorphism for all $X \in$* Ob($\mathscr{A}$'), *and assume both F and G are way-out right. Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D^{+}_{\mathscr{A}'}(\mathscr{A})$).

(3) *Assume there exists a subcollection $\mathscr{I}$ of* Ob($\mathscr{A}$') *such that* (a) *for any $X \in$* Ob($\mathscr{A}$') *there exists a monomorphism $X \to I$ with $I \in \mathscr{I}$, and* (b) *$\eta(I)$ is an isomorphism for all $I \in \mathscr{I}$, and assume both F and G are way-out right. Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D^{+}_{\mathscr{A}'}(\mathscr{A})$).

(4) *Assume $\eta(X)$ is an isomorphism for all $X \in$* Ob($\mathscr{A}$'), *and assume both F and G are way-out left. Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D^{-}_{\mathscr{A}'}(\mathscr{A})$).

(5) *Assume there exists a subcollection $\mathscr{P}$ of* Ob($\mathscr{A}$') *such that* (a) *for any $X \in$* Ob($\mathscr{A}$') *there exists an epimorphism $P \to X$ with $P \in \mathscr{P}$, and* (b) *$\eta(P)$ is an isomorphism for all $P \in \mathscr{P}$, and assume both F and G are way-out left. Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D^{-}_{\mathscr{A}'}(\mathscr{A})$).

(6) *Assume $\eta(X)$ is an isomorphism for all $X \in$* Ob($\mathscr{A}$'), *and assume both F and G are way-out in both directions. Then $\eta(X^\bullet)$ is an isomorphism for all $X^\bullet \in$* Ob($D_{\mathscr{A}'}(\mathscr{A})$).

Proof. We need the following.

Claim: Let $X^\bullet \in \mathrm{Ob}(D^*_{\mathscr{A}'}(\mathscr{A}))$. If $\eta(X^\bullet)$ is an isomorphism, so is $\eta(T^n X^\bullet)$ for all $n \in \mathbb{Z}$.

Proof. Let $F = (F, \alpha)$ and $G = (G, \beta)$. Then, since $\eta_T = \beta^{-1} \circ T\eta \circ \alpha$, it follows that $\eta(X^\bullet)$ is an isomorphism if and only if so is $\eta(T X^\bullet)$.

(1) Let $X^\bullet \in \mathrm{Ob}(D^b_{\mathscr{A}'}(\mathscr{A}))$. For $n \gg 0$, $\sigma_{>n}(X^\bullet) = 0$ and $\eta(\sigma_{>n}(X^\bullet))$ is an isomorphism. Let $n \in \mathbb{Z}$ and assume $\eta(\sigma_{>n}(X^\bullet))$ is an isomorphism. We claim that $\eta(\sigma_{\geq n}(X^\bullet))$ is an isomorphism. By Lemma 23.4 we have a triangle of the form

$$(T^{-n}(H^n(X^\bullet)), \sigma_{\geq n}(X^\bullet), \sigma_{>n}(X^\bullet), \cdot, \cdot, \cdot).$$

Since by Claim $\eta(T^{-n}(H^n(X^\bullet)))$ is an isomorphism, so is $\eta(\sigma_{\geq n}(X^\bullet))$ by Proposition 6.6. Thus, since $X^\bullet = \sigma_{>n}(X^\bullet)$ for $n \ll 0$, it follows by induction that $\eta(X^\bullet)$ is an isomorphism.

(2) Let $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$ and $n \in \mathbb{Z}$. We claim that $H^n(\eta(X^\bullet))$ is an isomorphism. Put $n_1 = n + 1$. There exist $n_2(F) \in \mathbb{Z}$ such that $H^i(F(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A})$ with $H^i(X^\bullet) = 0$ for $i < n_2(F)$, and $n_2(G) \in \mathbb{Z}$ such that $H^i(G(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A})$ with $H^i(X^\bullet) = 0$ for $i < n_2(G)$. Put $n_2 = \max\{n_2(F), n_2(G)\}$. Since $H^i(\sigma_{>n_2}(X^\bullet)) = 0$ for $i < n_2$, we have

$$H^n(F(\sigma_{>n_2}(X^\bullet))) = H^{n-1}(F(\sigma_{>n_2}(X^\bullet))) = 0,$$
$$H^n(G(\sigma_{>n_2}(X^\bullet))) = H^{n-1}(G(\sigma_{>n_2}(X^\bullet))) = 0.$$

Since by Lemma 23.4 we have a triangle of the form

$$(\sigma_{\leq n_2}(X^\bullet), X^\bullet, \sigma_{>n_2}(X^\bullet), \cdot, \cdot, \cdot),$$

we have a commutative diagram

$$\begin{array}{ccc} H^n(F(\sigma_{\leq n_2}(X^\bullet))) & \xrightarrow{\sim} & H^n(F((X^\bullet)) \\ H^n(\eta(\sigma_{\leq n_2}(X^\bullet)))\downarrow & & \downarrow H^n(\eta(X^\bullet)) \\ H^n(G(\sigma_{\leq n_2}(X^\bullet))) & \xrightarrow{\sim} & H^n(G((X^\bullet)) . \end{array}$$

By the part (1) $\eta(\sigma_{\leq n_2}(X^\bullet))$ is an isomorphism, so is $H^n(\eta(X^\bullet))$.

(3) By the part (2), it suffices to show that $\eta(X)$ is an isomorphism for all $X \in \mathrm{Ob}(\mathscr{A}')$. Let $X \in \mathrm{Ob}(\mathscr{A}')$. By hypothesis (a) X has a right resolution $X \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. It

suffices to show that $\eta(I^\bullet)$ is an isomorphism, the proof of which consists of two steps.

Step 1: $\eta(I^\bullet)$ is an isomorphism for all $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$.

Proof. Let $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$. For $n \gg 0$, since $\tau_{>n}(I^\bullet) = 0$, $\eta(\tau_{>n}(I^\bullet))$ is an isomorphism. Let $n \in \mathbb{Z}$ and assume $\eta(\tau_{>n}(I^\bullet))$ is an isomorphism. We claim that $\eta(\tau_{\geq n}(I^\bullet))$ is also an isomorphism. By Lemma 23.4 we have a triangle of the form

$$(\tau_{>n}(I^\bullet), \tau_{\geq n}(I^\bullet), T^{-n}(I^n), \cdot, \cdot, \cdot).$$

Since by Claim $\eta(T^{-n}(I^n))$ is an isomorphism, so is $\eta(\tau_{\geq n}(I^\bullet))$ by Proposition 6.6. Thus, since $I^\bullet = \tau_{>n}(I^\bullet)$ for $n \ll 0$, it follows by induction that $\eta(I^\bullet)$ is an isomorphism.

Step 2: $\eta(I^\bullet)$ is an isomorphism for all $I^\bullet \in \mathrm{Ob}(K^{+}(\mathscr{I}))$.

Proof. Let $I^\bullet \in \mathrm{Ob}(K^{+}(\mathscr{I}))$ and $n \in \mathbb{Z}$. We claim that $H^n(\eta(I^\bullet))$ is an isomorphism. Put $n_1 = n + 1$. There exist $n_2(F) \in \mathbb{Z}$ such that $H^i(F(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2(F)$, and $n_2(G) \in \mathbb{Z}$ such that $H^i(G(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2(G)$. Put $n_2 = \max\{n_2(F), n_2(G)\}$. Since $H^i(\tau_{\geq n}(I^\bullet)) = 0$ for $i < n_2$, we have

$$H^n(F(\tau_{\geq n}(I^\bullet))) = H^{n-1}(F(\tau_{\geq n}(I^\bullet))) = 0,$$
$$H^n(G(\tau_{\geq n}(I^\bullet))) = H^{n-1}(G(\tau_{\geq n}(I^\bullet))) = 0.$$

Since by Lemma 23.4 we have a triangle of the form

$$(\tau_{>n}(I^\bullet), I^\bullet, \tau_{\geq n}(I^\bullet), \cdot, \cdot, \cdot),$$

we have a commutative diagram

$$\begin{array}{ccc} & H^n(F(\tau_{>n}(I^\bullet))) \xrightarrow{\sim} H^n(F((I^\bullet)) & \\ H^n(\eta(\tau_{>n}(I^\bullet))) \downarrow & & \downarrow H^n(\eta(I^\bullet)) \\ & H^n(G(\tau_{>n}(I^\bullet))) \xrightarrow{\sim} H^n(G((I^\bullet)) . & \end{array}$$

By Step 1 $\eta(\tau_{>n}(I^\bullet))$ is an isomorphism, so is $H^n(\eta(I^\bullet))$.

(4) Dual of (2).

(5) Dual of (3).

(6) Let $X^\bullet \in \mathrm{Ob}(D_{\mathscr{A}'}(\mathscr{A}))$. By the part (2) $\eta(\sigma_{>0}(X^\bullet))$ is an isomorphism, and by the part

(4) $\eta(\sigma_{\leq 0}(X^\bullet))$ is an isomorphism. Since by Lemma 23.4 we have a triangle of the form

$$(\sigma_{\leq 0}(X^\bullet),\ X^\bullet,\ \sigma_{>0}(X^\bullet),\ \cdot,\ \cdot,\ \cdot),$$

it follows by Proposition 6.6 that $\eta(X^\bullet)$ is an isomorphism.

Proposition 23.6. *Let* $F: D^*_{\mathscr{A}'}(\mathscr{A}) \to D(\mathscr{B})$ *be a* ∂*-functor, where* $* = +, -,$ b *or nothing. Then the following hold.*

(1) *Assume* $F(X) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X \in \mathrm{Ob}(\mathscr{A}')$. *Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A}))$.

(2) *Assume* $F(X) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X \in \mathrm{Ob}(\mathscr{A}')$, *and assume F is way-out right. Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$.

(3) *Assume there exists a subcollection* $\mathscr{I}$ *of* $\mathrm{Ob}(\mathscr{A}')$ *such that* (a) *for any* $X \in \mathrm{Ob}(\mathscr{A}')$ *there exists a monomorphism* $X \to I$ *with* $I \in \mathscr{I}$, *and* (b) $F(I) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $I \in \mathscr{I}$, *and assume F is way-out right. Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$.

(4) *Assume* $F(X) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X \in \mathrm{Ob}(\mathscr{A}')$, *and assume F is way-out left. Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D^-_{\mathscr{A}'}(\mathscr{A}))$.

(5) *Assume there exists a subcollection* $\mathscr{P}$ *of* $\mathrm{Ob}(\mathscr{A}')$ *such that* (a) *for any* $X \in \mathrm{Ob}(\mathscr{A}')$ *there exists an epimorphism* $P \to X$ *with* $P \in \mathscr{P}$, *and* (b) $F(P) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $P \in \mathscr{P}$, *and assume F is way-out left. Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$.

(6) *Assume* $F(X) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X \in \mathrm{Ob}(\mathscr{A}')$, *and assume F is way-out in both directions. Then* $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ *for all* $X^\bullet \in \mathrm{Ob}(D_{\mathscr{A}'}(\mathscr{A}))$.

Proof. (1) Note first that $F(T^n X) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ for all $X \in \mathrm{Ob}(\mathscr{A}')$ and $n \in \mathbb{Z}$. Let $X^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A}))$. For $n \gg 0$, since $\sigma_{>n}(X^\bullet) = 0$, $F(\sigma_{>n}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. Let $n \in \mathbb{Z}$ and assume $F(\sigma_{>n}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. We claim that $F(\sigma_{\geq n}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. By Lemma 23.4 we have a triangle of the form

$$(T^{-n}(H^n(X^\bullet)),\ \sigma_{\geq n}(X^\bullet),\ \sigma_{>n}(X^\bullet),\ \cdot,\ \cdot,\ \cdot).$$

Since $F(T^{-n}(H^n(X^\bullet))) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, it follows that $F(\sigma_{\geq n}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. Thus, since $X^\bullet = \sigma_{>n}(X^\bullet)$ for $n \ll 0$, it follows by induction that $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$.

(2) Let $X^\bullet \in \mathrm{Ob}(D^+_{\mathscr{A}'}(\mathscr{A}))$ and $n \in \mathbb{Z}$. We claim $H^n(F(X^\bullet)) \in \mathrm{Ob}(\mathscr{B}')$. Put $n_1 = n + 1$. There exists $n_2 \in \mathbb{Z}$ such that $H^i(F(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2$. Since $H^i(\sigma_{>n_2}(X^\bullet)) = 0$ for $i < n_2$, we have

$$H^n(F(\sigma_{>n_2}(X^\bullet))) = H^{n-1}(F(\sigma_{>n_2}(X^\bullet))) = 0.$$

Thus, since by Lemma 23.4 we have a triangle of the form

$$(\sigma_{\le n_2}(X^\bullet),\ X^\bullet,\ \sigma_{>n_2}(X^\bullet),\ \cdot,\ \cdot,\ \cdot),$$

we get $H^n(F(\sigma_{\le n_2}(X^\bullet))) \cong H^n(F((X^\bullet))$. Since $\sigma_{\le n_2}(X^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A}))$, by the part (1) we have $F(\sigma_{\le n_2}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, so that $H^n(F((X^\bullet)) \cong H^n(F(\sigma_{\le n_2}(X^\bullet))) \in \mathrm{Ob}(\mathscr{B}')$.

(3) By the part (2), it suffices to show that $F(X) \in \mathrm{Ob}(\mathscr{B}')$ for all $X \in \mathrm{Ob}(\mathscr{A}')$. Let $X \in \mathrm{Ob}(\mathscr{A}')$. By hypothesis (a) X has a right resolution $X \to I^\bullet$ with $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$. It suffices to show that $F(I^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, the proof of which consists of two steps.

Step 1: $F(I^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ for all $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$.

Proof. Note first that $F(T^n I) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ for all $I \in \mathscr{I}$ and $n \in \mathbb{Z}$. Let $I^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathscr{I}))$. For $n \gg 0$, since $\tau_{>n}(I^\bullet) = 0$, $F(\tau_{>n}(I^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. Let $n \in \mathbb{Z}$ and assume $F(\tau_{>n}(I^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. We claim $F(\tau_{\ge n}(I^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. By Lemma 23.4 we have a triangle of the form

$$(\tau_{>n}(I^\bullet),\ \tau_{\ge n}(I^\bullet),\ T^{-n}(I^n),\ \cdot,\ \cdot,\ \cdot).$$

Since $F(T^{-n}(I^n)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, it follows that $F(\tau_{\ge n}(I^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. Thus, since $I^\bullet = \tau_{>n}(I^\bullet)$ for $n \ll 0$, it follows by induction that $F(I^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$.

Step 2: $F(I^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$ for all $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$.

Proof. Let $I^\bullet \in \mathrm{Ob}(K^+(\mathscr{I}))$ and $n \in \mathbb{Z}$. We claim $H^n(F(I^\bullet)) \in \mathrm{Ob}(\mathscr{B}')$. Put $n_1 = n + 1$. There exists $n_2 \in \mathbb{Z}$ such that $H^i(F(X^\bullet)) = 0$ for $i < n_1$ and $X^\bullet \in \mathrm{Ob}(D(\mathscr{A}))$ with $H^i(X^\bullet) = 0$ for $i < n_2$. Since $H^i(\sigma_{>n_2}(I^\bullet)) = 0$ for $i < n_2$, we have

$$H^n(F(\sigma_{>n_2}(I^\bullet))) = H^{n-1}(F(\sigma_{>n_2}(I^\bullet))) = 0.$$

Thus, since by Lemma 23.4 we have a triangle of the form

$$(\sigma_{\le n_2}(I^\bullet),\ I^\bullet,\ \sigma_{>n_2}(I^\bullet),\ \cdot,\ \cdot,\ \cdot),$$

we get $H^n(F(\sigma_{\le n_2}(I^\bullet))) \cong H^n(F((I^\bullet))$. Since $\sigma_{\le n_2}(I^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}_{\mathscr{A}'}(\mathscr{A}))$, by the part (1) we have $F(\sigma_{\le n_2}(I^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, so that $H^n(F((I^\bullet)) \cong H^n(F(\sigma_{\le n_2}(I^\bullet))) \in \mathrm{Ob}(\mathscr{B}')$.

(4) Dual of (2).

(5) Dual of (3).

(6) Let $X^\bullet \in \mathrm{Ob}(D_{\mathscr{A}'}(\mathscr{A}))$. By the part (2) $F(\sigma_{>0}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$, and by the part (4) $F(\sigma_{\le 0}(X^\bullet)) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$. Thus, since by Lemma 23.4 we have a triangle of the form

$$(\sigma_{\leq 0}(X^\bullet),\ X^\bullet,\ \sigma_{>0}(X^\bullet),\ \cdot,\ \cdot,\ \cdot),$$

it follows that $F(X^\bullet) \in \mathrm{Ob}(D_{\mathscr{B}'}(\mathscr{B}))$.

§24. Connections between $\boldsymbol{R}\mathrm{Hom}^\bullet$ and $\overset{\boldsymbol{L}}{\otimes}$

Throughout this section, R is a commutative ring and A, B are R-algebras. For any ring A, we denote by $K(\mathrm{Inj}\ A)_\mathrm{L}$ (resp. $K(\mathrm{Proj}\ A))_\mathrm{L}$) the full subcategory of $K(\mathrm{Inj}\ A)$ (resp. $K(\mathrm{Proj}\ A)$)) consisting of $\mathcal{U}$-local (resp $\mathcal{U}$-colocal) complexes in $K(\mathrm{Inj}\ A)$ (resp. $K(\mathrm{Proj}\ A)$)), where $\mathcal{U}$ is the épaisse subcategory of $K(\mathrm{Mod}\ A)$ consisting of acyclic complexes. Also, we denote by E an injective cogenerator in $\mathrm{Mod}\ R$ and by D both $\mathrm{Hom}_R(-, E)$ and $\boldsymbol{R}\mathrm{Hom}^\bullet(-, E)$.

Proposition 24.1. (1) *There exists a natural isomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(M^\bullet \overset{\boldsymbol{L}}{\otimes} V^\bullet,\ N^\bullet) \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(M^\bullet, \boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet,\ N^\bullet))$$

for $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A^{\mathrm{op}}))$, $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$ *and* $N^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ B^{\mathrm{op}}))$. *In particular, for any* $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$,

$$- \overset{\boldsymbol{L}}{\otimes} V^\bullet : D(\mathrm{Mod}\ A^{\mathrm{op}}) \to D(\mathrm{Mod}\ B^{\mathrm{op}})$$

is a left adjoint of

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, -) : D(\mathrm{Mod}\ B^{\mathrm{op}}) \to D(\mathrm{Mod}\ A^{\mathrm{op}}).$$

(2) *There exists a natural isomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet,\ Y^\bullet) \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, \boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet,\ Y^\bullet))$$

for $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ B))$, $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$ *and* $Y^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A))$. *In particular, for any* $V^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$,

$$V^\bullet \overset{\boldsymbol{L}}{\otimes} - : D(\mathrm{Mod}\ B) \to D(\mathrm{Mod}\ A)$$

is a left adjoint of

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, -) : D(\mathrm{Mod}\ A) \to D(\mathrm{Mod}\ B).$$

Proof. (1) By Propositions 12.21 and 12.16 we may assume $M^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\ A^{\mathrm{op}})_\mathrm{L})$ and $N^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\ B^{\mathrm{op}})_\mathrm{L})$, respectively. Then by Lemma 19.5 we have canonical isomorphisms

$$\boldsymbol{R}\mathrm{Hom}^\bullet(M^\bullet \overset{\boldsymbol{L}}{\otimes} V^\bullet,\ N^\bullet) \cong \boldsymbol{R}\mathrm{Hom}^\bullet(M^\bullet \otimes V^\bullet,\ N^\bullet)$$

$$
\begin{aligned}
&\cong \operatorname{Hom}^\bullet(M^\bullet \otimes V^\bullet, N^\bullet)\\
&\cong \operatorname{Hom}^\bullet(M^\bullet, \operatorname{Hom}^\bullet(V^\bullet, N^\bullet))\\
&\cong \boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet, \operatorname{Hom}^\bullet(V^\bullet, N^\bullet))\\
&\cong \boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(V^\bullet, N^\bullet)).
\end{aligned}
$$

Next, by applying $H^0 : D(\text{Mod } R) \to \text{Mod } R$, we get natural isomorphisms

$$
\begin{aligned}
\operatorname{Hom}_{D(\text{Mod } B^{\text{op}})}(M^\bullet \overset{L}{\otimes} V^\bullet, N^\bullet) &\cong H^0(\boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet \otimes V^\bullet, N^\bullet))\\
&\cong H^0(\boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(V^\bullet, N^\bullet)))\\
&\cong \operatorname{Hom}_{D(\text{Mod } A^{\text{op}})}(M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(V^\bullet, N^\bullet))
\end{aligned}
$$

for $M^\bullet \in \text{Ob}(D(\text{Mod } A^{\text{op}}))$ and $N^\bullet \in \text{Ob}(D(\text{Mod } B^{\text{op}}))$.

(2) Similar to (1).

Definition 24.1. Let $V \in \text{Mod } A \otimes_R B^{\text{op}}$. Then there exist a natural homomorphism

$$\phi_{X,Y} : \operatorname{Hom}_A(X, V) \otimes_B Y \to \operatorname{Hom}_A(X, V \otimes_B Y),\ h \otimes y \mapsto (x \mapsto h(x) \otimes y),$$

for $X \in \text{Mod } A$ and $Y \in \text{Mod } B$, and a natural homomorphism

$$\psi_{M,X} : \operatorname{Hom}_B(V, M) \otimes_A X \to \operatorname{Hom}_B(\operatorname{Hom}_A(X, V), M),\ h \otimes x \mapsto (f \mapsto h(f(x))),$$

for $M \in \text{Mod } B^{\text{op}}$ and $X \in \text{Mod } A$.

Lemma 24.2. *Let* $V \in \text{Mod } A \otimes_R B^{\text{op}}$. *Then the following hold*.

(1) $\phi_{X,Y}$ *is an isomorphism for all* $X \in \text{mod } A$ *and* $Y \in \text{Flat } B$.

(2) $\phi_{X,Y}$ *is an isomorphism for all* $X \in \text{Proj } A$ *and* $Y \in \text{mod } B$.

(3) $\psi_{M,X}$ *is an isomorphism for all* $M \in \text{Inj } B^{\text{op}}$ *and* $X \in \text{mod } A$.

Proof. Straightforward.

Lemma 24.3. *There exists a natural homomorphism*

$$\operatorname{Hom}^\bullet(X^\bullet, V^\bullet) \otimes Y^\bullet \to \operatorname{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet)$$

for $X^\bullet \in \text{Ob}(K^-(\text{Mod } A))$, $V^\bullet \in \text{Ob}(K^+(\text{Mod } A \otimes_R B^{\text{op}}))$ *and* $Y^\bullet \in \text{Ob}(K^+(\text{Mod } B))$, *which is an isomorphism provided either*

(a) $X^\bullet \in \text{Ob}(K^-(\text{mod } A))$ *and* $Y^\bullet \in \text{Ob}(K^+(\text{Flat } B))$, *or*

(b) $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{Proj}\, A))$ *and* $Y^\bullet \in \mathrm{Ob}(K^+(\mathrm{mod}\, B))$.

Proof. For any $n \in \mathbb{Z}$ we may consider that

$$[\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \otimes Y^\bullet]^n = \bigoplus_{p+q+r=n} \mathrm{Hom}_A(X^{-p}, V^q) \otimes_B Y^r,$$

$$[\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet)]^n = \bigoplus_{p+q+r=n} \mathrm{Hom}_A(X^{-p}, V^q \otimes_B Y^r),$$

which are finite direct sums, and we have a homomorphism

$$\phi^n_{X,Y} : \bigoplus_{p+q+r=n} \mathrm{Hom}_A(X^{-p}, V^q) \otimes_B Y^r \to \bigoplus_{p+q+r=n} \mathrm{Hom}_A(X^{-p}, V^q \otimes_B Y^r)$$

such that

$$\phi^n_{X,Y}(h^{p,q} \otimes y^r)(x^p) = (-1)^{pr}\, h^{p,q}(x^p) \otimes y^r$$

for $h^{p,q} \in \mathrm{Hom}_A(X^{-p}, V^q)$, $y^r \in Y^r$ and $x^p \in X^{-p}$, where $p, q, r \in \mathbb{Z}$ with $p + q + r = n$. It is easy to see that ϕ commutes with differentials. The remaining assertions follow by Lemma 24.2.

Lemma 24.4. *There exists a natural homomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \overset{\boldsymbol{L}}{\otimes} Y^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\, A))$, $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *and* $Y^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\, B)_{\mathrm{fTd}})$, *which is an isomorphism provided either*

(a) *A is left coherent and* $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A))$, *or*

(b) *B is left coherent and* $Y^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, B)_{\mathrm{fpd}})$.

Proof. By Propositions 10.15 and 22.17 we may assume $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{Proj}\, A))$ and $Y^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Flat}\, B))$, respectively. Then we have

$$\begin{aligned}\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet &\cong \mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \\ &\cong \mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \otimes Y^\bullet,\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \overset{\boldsymbol{L}}{\otimes} Y^\bullet) &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet) \\ &\cong \mathrm{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet).\end{aligned}$$

Thus by Lemma 24.3 we get a desired homomorphism

(a) Assume A is left coherent and $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A))$. Then by Proposition 10.15 we may assume $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{proj}\ A))$. Thus by Lemma 24.3(1)

$$\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \otimes Y^\bullet \xrightarrow{\sim} \mathrm{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet).$$

(b) Assume B is left coherent, $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ A))$ and $Y^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ B)_{\mathrm{fpd}})$. Then by Proposition 11.17 we may assume $Y^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\ B))$. Thus by Lemma 24.3(2)

$$\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \otimes Y^\bullet \xrightarrow{\sim} \mathrm{Hom}^\bullet(X^\bullet, V^\bullet \otimes Y^\bullet).$$

Proposition 24.5. *Let A be left coherent. Then there exists a natural isomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \overset{\boldsymbol{L}}{\otimes} Y^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D_{\mathrm{c}}^-(\mathrm{Mod}\ A))$, $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$ *and* $Y^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ B)_{\mathrm{fTd}})$.

Proof. Let $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$ and $Y^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ B)_{\mathrm{fTd}})$. Then by Lemma 24.4 there exists a natural homomorphism

$$\phi_X : \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet \overset{\boldsymbol{L}}{\otimes} Y^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ A))$. By Proposition 22.7(1) $\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet \overset{\boldsymbol{L}}{\otimes} Y^\bullet)$ is way-out right, and by Propositions 22.7(1) and 22.17 $(- \overset{\boldsymbol{L}}{\otimes} Y^\bullet) \circ \boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet)$ is also way-out right. Thus, since by Lemma 24.4(1) ϕ_X is an isomorphism for all $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A))$, it follows by Proposition 23.5(2) that ϕ_X is an isomorphism for all $X^\bullet \in \mathrm{Ob}(D_{\mathrm{c}}^-(\mathrm{Mod}\ A))$.

Lemma 24.6. (1) *There exists a natural homomorphism*

$$\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \otimes X^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

for $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Mod}\ B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\ A))$, *which is an isomorphism provided* $M^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Inj}\ B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{mod}\ A))$.

(2) *There exists a natural homomorphism*

$$\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \otimes X^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

for $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\ A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\ B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\ A))$, *which is an isomorphism provided* $M^\bullet \in \mathrm{Ob}(K^+(\mathrm{Inj}\ B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K^+(\mathrm{mod}\ A))$.

Proof. Let $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, B^{\mathrm{op}}))$ and assume either (1) $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Mod}\, B^{\mathrm{op}}))$, or (2) $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}})$. Let $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$. For any $n \in \mathbb{Z}$ we may consider that

$$[\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \otimes X^\bullet]^n = \bigoplus_{p+q+r=n} \mathrm{Hom}_B(V^{-p}, M^q) \otimes_A X^r,$$

$$[\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)]^n = \prod_{p+q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^r, V^{-p}), M^q).$$

We have a homomorphism

$$\psi^n_{M,X} : \bigoplus_{p+q+r=n} \mathrm{Hom}_B(V^{-p}, M^q) \otimes_A X^r \to \prod_{p+q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^r, V^{-p}), M^q)$$

such that

$$\psi^n_{M,X}(h^{p,q} \otimes x^r)(f^{-r,-p}) = (-1)^{\frac{r(2p+r+1)}{2}} h^{p,q}(f^{-r,-p}(x^r))$$

for $h^{p,q} \in \mathrm{Hom}_B(V^{-p}, I^q)$, $x^r \in X^r$ and $f^{-r,-p} \in \mathrm{Hom}_A(X^r, V^{-p})$, where $p, q, r \in \mathbb{Z}$ with $p + q + r = n$. It is easy to see that ψ commutes with differentials. Next, assume either (1) $M^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Inj}\, B^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{mod}\, A))$, or (2) $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(K^+(\mathrm{Inj}\, B^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(K^+(\mathrm{mod}\, A))$. Then

$$\bigoplus_{p+q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^r, V^{-p}), M^q) = \prod_{p+q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^r, V^{-p}), M^q)$$

for all $n \in \mathbb{Z}$ and by Lemma 24.2(3) $\psi_{M,X}$ is an isomorphism.

Lemma 24.7. (1) *There exists a natural homomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{L}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

for $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\, B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A))$, *which is an isomorphism if A is left coherent,* $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$ *and* $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A))$.

(2) *There exists a natural homomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{L}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

for $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A))$, *which is* *an isomorphism if A is left coherent,* $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A)_{\mathrm{fpd}})$.

Proof. Let $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ and $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, B^{\mathrm{op}}))$. Assume either (1) $M^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\, B^{\mathrm{op}}))$, or (2) $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}})$. By Proposition 12.16 we may assume $M^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\, B^{\mathrm{op}})_{\mathrm{L}})$. Furthermore, in case $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}}))$, by Proposition 10.13 we may assume $M^\bullet \in \mathrm{Ob}(K^+(\mathrm{Inj}\, B^{\mathrm{op}}))$. Let $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\, A))$. By Proposition 12.21 we may assume $X^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\, A)_{\mathrm{L}})$. Since we have isomorphisms

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet \cong \mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \otimes X^\bullet,$$
$$\boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet) \cong \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet),$$

by Lemma 24.6 we get desired homomorphisms.

Next, assume A is left coherent. Assume either (1) $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$ and $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A))$, or (2) $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A)_{\mathrm{fpd}})$. By Proposition 10.13 we may assume $M^\bullet \in \mathrm{Ob}(K^+(\mathrm{Inj}\, B^{\mathrm{op}}))$. In case $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$, by Proposition 11.13 we may assume $M^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Inj}\, B^{\mathrm{op}}))$. Also, by Propodition 10.15 we may assume $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{proj}\, A))$. Furthermore, in case $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A)_{\mathrm{fpd}})$, by Proposition 11.17 we may assume $X^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, A))$. Then, since by Lemma 24.6 the canonical homomorphism

$$\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \otimes X^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

is an isomorphism, so is

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet).$$

Proposition 24.8. *Let A be left coherent. Then there exists a natural isomorphism*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet \tilde{\to} \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), M^\bullet)$$

for $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$, $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$ *and* $X^\bullet \in \mathrm{Ob}(D_{\mathrm{c}}^-(\mathrm{Mod}\, A))$.

Proof. Let $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ and $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$. By Proposition 11.13 we may assume $M^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Inj}\, B^{\mathrm{op}}))$. Then $\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \cong \mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \in \mathrm{Ob}(D^-(\mathrm{Mod}\, A^{\mathrm{op}}))$ and by Proposition 22.15 $\boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet, M^\bullet) \overset{\boldsymbol{L}}{\otimes} -$ is way-out left. Also, by Propositions 22.7(1) and 22.8 $\boldsymbol{R}\mathrm{Hom}^\bullet(-, M^\bullet) \circ \boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet)$ is way-out left. By Lemma 24.7(1) there exists a natural homomorphism

$$\psi_X : \boldsymbol{R}\mathrm{Hom}^\bullet(V^\bullet,\ M^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ V^\bullet),\ M^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{Mod}\ A))$, which is an isomorphism for $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A))$. Thus, it follows by Proposition 23.5(4) that ψ_X is an isomorphism for $X^\bullet \in \mathrm{Ob}(D^-_{\mathrm{c}}(\mathrm{Mod}\ A))$.

Corollary 24.9. *Let A be left coherent. Then for any* $X^\bullet \in \mathrm{Ob}(D^-_{\mathrm{c}}(\mathrm{Mod}\ A))$, $Y^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A))$ *and* $i \in \mathbb{Z}$ *there exists an isomorphism*

$$D(\mathrm{Ext}^i(X^\bullet,\ Y^\bullet)) \cong \mathrm{Tor}_i(D(Y^\bullet),\ X^\bullet).$$

In particular, $Y^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A)_{\mathrm{fid}})$ *if and only if* $D(Y^\bullet) \in \mathrm{Ob}(D^-(\mathrm{Mod}\ A^{\mathrm{op}})_{\mathrm{fTd}})$.

Proof. It follows by Proposition 24.8 that $D(Y^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet \cong D(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ Y^\bullet))$. Thus, for any $i \in \mathbb{Z}$, we have

$$\begin{aligned} D(\mathrm{Ext}^i(X^\bullet,\ Y^\bullet)) &\cong D(H^i(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ V^\bullet))) \\ &\cong H^{-i}(D(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ V^\bullet))) \\ &\cong H^{-i}(D(Y^\bullet) \overset{\boldsymbol{L}}{\otimes} X^\bullet) \\ &\cong \mathrm{Tor}_i(D(Y^\bullet),\ X^\bullet). \end{aligned}$$

The last assertion follows by Lemmas 20.7 and 20.8.

Proposition 24.10. *Let A be left coherent. Then there exists a natural isomorphism*

$$D(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ Y^\bullet)) \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(Y^\bullet, D(A) \overset{\boldsymbol{L}}{\otimes} X^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D^-_{\mathrm{c}}(\mathrm{Mod}\ A)_{\mathrm{fTd}})$ *and* $Y^\bullet \in \mathrm{Ob}(D^-_{\mathrm{c}}(\mathrm{Mod}\ A))$.

Proof. Since $\mathrm{Hom}_R(-,\ E) \cong \mathrm{Hom}_A(-,\ D(A))$ as a functor from Mod A to Mod A^{op}, it follows that $\boldsymbol{R}\mathrm{Hom}^\bullet(-,\ E) \cong \boldsymbol{R}\mathrm{Hom}^\bullet(-,\ D(A))$ as a ∂-functor from $D(\mathrm{Mod}\ A)$ to $D(\mathrm{Mod}\ A^{\mathrm{op}})$. Thus by Propositions 24.5 and 24.8 we have

$$\begin{aligned} D(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ Y^\bullet)) &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\ Y^\bullet), E) \\ &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(Y^\bullet, E) \overset{\boldsymbol{L}}{\otimes} X^\bullet \\ &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(Y^\bullet, D(A)) \overset{\boldsymbol{L}}{\otimes} X^\bullet \\ &\cong \boldsymbol{R}\mathrm{Hom}^\bullet(Y^\bullet, D(A) \overset{\boldsymbol{L}}{\otimes} X^\bullet). \end{aligned}$$

Definition 24.2. There exists a natural homomorphism

$$\psi_{M,X} : M \otimes_A X \to \mathrm{Hom}_A(\mathrm{Hom}_A(X, A), M),\ m \otimes x \mapsto (f \mapsto mf(x)),$$

for $M \in \mathrm{Mod}\, B^{\mathrm{op}}$ and $X \in \mathrm{Mod}\, A$.

Lemma 24.11. *$\psi_{M,X}$ is an isomorphism provided $X \in \mathrm{proj}\, A$.*

Proof. Straightforward.

Lemma 24.12. *There exists a natural homomorphism*

$$M^\bullet \otimes X^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A), M^\bullet)$$

for $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$, *which is an isomorphism provided either*

(a) $X^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, A))$, *or*

(b) $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Mod}\, A^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{proj}\, A))$.

Proof. Let $M^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$. For any $n \in \mathbb{Z}$ we may consider that

$$[M^\bullet \otimes X^\bullet]^n = \bigoplus_{p+q=n} M^p \otimes_A X^q,$$

$$[\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A), M^\bullet)]^n = \prod_{p+q=n} \mathrm{Hom}_A(\mathrm{Hom}_A(X^q, A), M^p).$$

For any $p, q \in \mathbb{Z}$ we have a homomorphism

$$\phi^{p,q} : M^p \otimes_A X^q \to \mathrm{Hom}_A(\mathrm{Hom}_A(X^q, A), M^p),\ m \otimes x \mapsto (h \mapsto (-1)^q mh(x)),$$

which is an isomorphism if $X^q \in \mathrm{proj}\, A$. Thus for any $n \in \mathbb{Z}$ we have a homomorphism

$$\phi^n : \bigoplus_{p+q=n} M^p \otimes_A X^q \to \prod_{p+q=n} \mathrm{Hom}_A(\mathrm{Hom}_A(X^q, A), M^p).$$

It is easy to see that ϕ commutes with differentials. Assume either (a) $X^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, A))$, or (b) $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{Mod}\, A^{\mathrm{op}}))$ and $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{proj}\, A))$. Then

$$\bigoplus_{p+q=n} \mathrm{Hom}_A(\mathrm{Hom}_A(X^q, A), M^p) = \prod_{p+q=n} \mathrm{Hom}_A(\mathrm{Hom}_A(X^q, A), M^p)$$

for all $n \in \mathbb{Z}$. Thus, since by Lemma 24.11 $\phi^{p,q}$ is an isomorphism for all $p, q \in \mathbb{Z}$, it follows that ϕ is an isomorphism.

Lemma 24.13. *There exists a natural homomorphism*

$$M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet)$$

for $M^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A^{\mathrm{op}}))$ *and* $X^\bullet \in \mathrm{Ob}(D(\mathrm{Mod}\ A))$, *which is an isomorphism provided A is left coherent and either*

(a) $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A)_{\mathrm{fpd}})$, *or*

(b) $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A^{\mathrm{op}})_{\mathrm{fid}})$ *and* $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A))$.

Proof. By Propositions 12.16 and 12.21 we may assume $M^\bullet \in \mathrm{Ob}(K(\mathrm{Inj}\ A^{\mathrm{op}})_{\mathrm{L}})$ and $X^\bullet \in \mathrm{Ob}(K(\mathrm{Proj}\ A)_{\mathrm{L}})$, respectively. Then the canonical homomorphisms

$$M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet \to M^\bullet \otimes X^\bullet$$

$$\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet) \to \boldsymbol{R}\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet)$$

are isomorphisms. Also, since the canonical homomorphism

$$\mathrm{Hom}^\bullet(X^\bullet, A) \to \boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, A)$$

is an isomorphism, we have a natural isomorphism

$$\boldsymbol{R}\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet) \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet).$$

Thus by Lemma 24.12 we get a natural homomorphism

$$M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet \to \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet).$$

Now, assume A is left coherent. Assume either (a) $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A)_{\mathrm{fpd}})$ or (b) $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A^{\mathrm{op}})_{\mathrm{fid}})$ and $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A))$. By Proposition 10.15 we may assume $X^\bullet \in \mathrm{Ob}(K^-(\mathrm{proj}\ A))$. Furthermore, in case $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\ A)_{\mathrm{fpd}})$, by Proposition 11.17 we may assume $X^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\ A))$. Also, in case $M^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\ A^{\mathrm{op}})_{\mathrm{fid}})$, by Proposition 10.13 we may assume $M^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Inj}\ A^{\mathrm{op}}))$. Then, since by Lemma 24.12 the canonical homomorphism

$$M^\bullet \otimes X^\bullet \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, A),\ M^\bullet)$$

is an isomorphism, so is

$$M^\bullet \overset{L}{\otimes} X^\bullet \to \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, A),\ M^\bullet).$$

Proposition 24.14. *Let A be left coherent. Then there exists a natural isomorphism*

$$M^\bullet \overset{L}{\otimes} X^\bullet \to \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, A),\ M^\bullet)$$

for $M^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fid}})$ *and* $X^\bullet \in \operatorname{Ob}(D_c^-(\operatorname{Mod} A))$.

Proof. Let $M^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fid}})$. Then by Proposition 11.13 we may assume $M^\bullet \in K^b(\operatorname{Inj} B^{\mathrm{op}})$, so that by Proposition 22.15 $M^\bullet \overset{L}{\otimes} -$ is way-out left. Also, by Propositions 22.7(1) and 22.8 $\boldsymbol{R}\operatorname{Hom}^\bullet(-,\ M^\bullet) \circ \boldsymbol{R}\operatorname{Hom}^\bullet(-, A)$ is way-out left. By Lemma 24.13 there exists a natural homomorphism

$$\psi_X : M^\bullet \overset{L}{\otimes} X^\bullet \to \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, A),\ M^\bullet)$$

for $X^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A))$, which is an isomorphism provided $X^\bullet \in \operatorname{Ob}(D^-(\operatorname{mod} A))$. Thus by Proposition 23.5(4) ψ_X is an isomorphism for $X^\bullet \in \operatorname{Ob}(D_c^-(\operatorname{Mod} A))$.

Corollary 24.15. *Let A be left coherent with* inj dim $A_A < \infty$. *Then there exists a natural isomorphism*

$$X^\bullet \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, A), A)$$

for $X^\bullet \in \operatorname{Ob}(D_c^-(\operatorname{Mod} A))$.

Proposition 24.16. *Let A be commutative. Then there exists a natural homomorphism*

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet,\ Y^\bullet \overset{L}{\otimes} Z^\bullet) \to \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet \overset{L}{\otimes} \boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet)$$

for $X^\bullet$, $Y^\bullet$ *and* $Z^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A))$, *which is an isomorphism provided A is coherent and either*

(a) $Z^\bullet \in \operatorname{Ob}(D^-(\operatorname{mod} A)_{\mathrm{fpd}})$, *or*
(b) $Y^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A)_{\mathrm{fid}})$ *and* $Z^\bullet \in \operatorname{Ob}(D^-(\operatorname{mod} A))$.

Proof. By Lemma 24.13 we have a natural homomorphism

$$Y^\bullet \overset{L}{\otimes} Z^\bullet \to \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet).$$

Thus we get a natural homomorphism

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet,\ Y^\bullet \overset{L}{\otimes} Z^\bullet) \to \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet)).$$

Also, by Proposition 24.1 we have a natural isomorphism

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet)) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet \overset{L}{\otimes} \boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet).$$

Consequently, we get a desired natural homomorphism

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet,\ Y^\bullet \overset{L}{\otimes} Z^\bullet) \to \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet \overset{L}{\otimes} \boldsymbol{R}\operatorname{Hom}^\bullet(Z^\bullet, A),\ Y^\bullet).$$

The last assertion follows by Lemma 24.13.

§25. Duality in Coherent rings

Throughout this section, R is a commutative ring and A, B are R-algebras. For any ring A, we denote by $K(\mathrm{Inj}\, A)_{\mathrm{L}}$ (resp. $K(\mathrm{Proj}\, A)_{\mathrm{L}}$) the full subcategory of $K(\mathrm{Inj}\, A)$ (resp. $K(\mathrm{Proj}\, A)$) consisting of $\mathcal{U}$-local (resp. $\mathcal{U}$-colocal) complexes, where $\mathcal{U}$ is the épaisse subcategory of $K(\mathrm{Mod}\, A)$ consisting of acyclic complexes.

Definition 25.1. Let $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$. Then, for any $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ and $n \in \mathbb{Z}$,

$$[\,\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), V^\bullet)]^n = \bigoplus_{p-q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^p, V^q), V^r),$$

which is a finite direct sum, and the differential is given by

$$\begin{aligned} d^n_{\mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), V^\bullet)}(h) = {} & (-1)^{r+1} h \circ \mathrm{Hom}_A(d_X^p, V^q) \\ & + (-1)^{n+1} h \circ \mathrm{Hom}_A(X^p, d_V^{q-1}) \\ & + d_V^r \circ h \end{aligned}$$

for $h \in \mathrm{Hom}_B(\mathrm{Hom}_A(X^p, V^q), V^r)$. For any $X^\bullet \in \mathrm{Ob}(K(\mathrm{Mod}\, A))$ and $n \in \mathbb{Z}$, we define a homomorphism

$$X^n \to \mathrm{Hom}_B(\mathrm{Hom}_A(X^n, V^r), V^r),\ \ x \mapsto (f \mapsto (-1)^{n(r+1)} f(x)),$$

for each $r \in \mathbb{Z}$ and set

$$\varepsilon_X^n : X^n \to \bigoplus_r \mathrm{Hom}_B(\mathrm{Hom}_A(X^n, V^r), V^r) \to \bigoplus_{p-q+r=n} \mathrm{Hom}_B(\mathrm{Hom}_A(X^p, V^q), V^r).$$

Then ε_X commutes with differentials. Thus we get a homomorphism of ∂-functors

$$\varepsilon : \mathbf{1}_{K(\mathrm{Mod}\, A)} \to \mathrm{Hom}^\bullet(-, V^\bullet) \circ \mathrm{Hom}^\bullet(-, V^\bullet)$$

Remark 25.1. Let $V^\bullet \in \mathrm{Ob}(C(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$. There exists a ring homomorphism

$$\varphi : A \to \mathrm{End}_{C(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet)$$

such that $\varphi(a)^n(v) = av$ for $a \in A$, $n \in \mathbb{Z}$ and $v \in V^n$, which gives rise to a ring homomorphism

$$A \to \mathrm{End}_{K(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet).$$

On the other hand, since we have homomorphisms in $C(\text{Mod}\,A)$

$$A \to \text{Hom}^\bullet(\text{Hom}^\bullet(A,\ V^\bullet),\ V^\bullet) \xrightarrow{\sim} \text{Hom}^\bullet(V^\bullet,\ V^\bullet),$$

by Lemma 18.3 we get a homomorphism in Mod A

$$A \to \text{End}_{K(\text{Mod}\,B^{\text{op}})}(V^\bullet),$$

which coinsides with the above ring homomorphism.

Lemma 25.1. *For any* $V^\bullet \in \text{Ob}(D^{\text{b}}(\text{Mod}\,A \otimes_R B^{\text{op}}))$ *there exists a homomorphism of* ∂*-functors*

$$\mathbf{1}_{D(\text{Mod}\,A)} \to \boldsymbol{R}\text{Hom}^\bullet(-,\ V^\bullet) \circ \boldsymbol{R}\text{Hom}^\bullet(-,\ V^\bullet).$$

Proof. By Proposition 12.21 $K(\text{Proj}\,A)_{\text{L}} \xrightarrow{\sim} D(\text{Mod}\,A)$. Let $P^\bullet \in \text{Ob}(K(\text{Proj}\,A)_{\text{L}})$. We have natural homomorphisms

$$P^\bullet \to \text{Hom}^\bullet(\text{Hom}^\bullet(P^\bullet,\ V^\bullet),\ V^\bullet),$$
$$\text{Hom}^\bullet(\text{Hom}^\bullet(P^\bullet,\ V^\bullet),\ V^\bullet) \to \boldsymbol{R}\text{Hom}^\bullet(\text{Hom}^\bullet(P^\bullet,\ V^\bullet),\ V^\bullet).$$

Also, since the canonical homomorphism

$$\text{Hom}^\bullet(P^\bullet,\ V^\bullet) \to \boldsymbol{R}\text{Hom}^\bullet(P^\bullet,\ V^\bullet)$$

is an isomorphism, we have a natural isomorphism

$$\boldsymbol{R}\text{Hom}^\bullet(\text{Hom}^\bullet(P^\bullet,\ V^\bullet),\ V^\bullet) \xrightarrow{\sim} \boldsymbol{R}\text{Hom}^\bullet(\boldsymbol{R}\text{Hom}^\bullet(P^\bullet,\ V^\bullet,\ V^\bullet).$$

Consequently, we get a desired natural homomorphism

$$P^\bullet \to \boldsymbol{R}\text{Hom}^\bullet(\boldsymbol{R}\text{Hom}^\bullet(P^\bullet,\ V^\bullet),\ V^\bullet).$$

Definition 25.2. Let $V^\bullet \in \text{Ob}(D^{\text{b}}(\text{Mod}\,A \otimes_R B^{\text{op}}))$. A complex $X^\bullet \in \text{Ob}(D(\text{Mod}\,A))$ is called $V^\bullet$-reflexive if the canonical homomorphism

$$X^\bullet \to \boldsymbol{R}\text{Hom}^\bullet(\boldsymbol{R}\text{Hom}^\bullet(X^\bullet,\ V^\bullet),\ V^\bullet).$$

is an isomorphism.

Remark 25.2. Let $V^\bullet \in \mathrm{Ob}(C^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$. There exists a sequence of homomorphisms in $D(\mathrm{Mod}\, A)$

$$A \to \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(A,\ V^\bullet),\ V^\bullet) \to \boldsymbol{R}\,\mathrm{Hom}^\bullet(\boldsymbol{R}\,\mathrm{Hom}^\bullet(A,\ V^\bullet),\ V^\bullet).$$

Thus, since $V^\bullet \cong \mathrm{Hom}^\bullet(A,\ V^\bullet) \cong \boldsymbol{R}\,\mathrm{Hom}^\bullet(A,\ V^\bullet)$, by Lemma 18.3 and Proposition 18.9(2) we get a sequence of homomorphisms in Mod A

$$A \to \mathrm{End}_{K(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet) \to \mathrm{End}_{D(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet),$$

which are canonical ring homomorphisms.

Lemma 25.2. *For* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *the following are equivalent.*

(1) *A is $V^\bullet$-reflexive.*

(2) (a) $\mathrm{Ext}^i(V^\bullet,\ V^\bullet) = 0$ *for* $i\ \ 0$ *in* $D(\mathrm{Mod}\, B^{\mathrm{op}})$, *and*

(b) *the canonical ring homomorphism* $A \to \mathrm{End}_{D(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet)$ *is an isomorphism.*

Proof. (1) $\Rightarrow$ (2). By Proposition 18.9(2)

$$\begin{aligned}\mathrm{Ext}^i(V^\bullet,\ V^\bullet) &\cong H^i(\boldsymbol{R}\,\mathrm{Hom}^\bullet(V^\bullet,\ V^\bullet))\\ &\cong H^i(A)\\ &= 0\end{aligned}$$

for $i\ \ 0$ and

$$\begin{aligned}A &= H^0(A)\\ &\cong H^0(\boldsymbol{R}\,\mathrm{Hom}^\bullet(V^\bullet,\ V^\bullet))\\ &\cong \mathrm{End}_{D(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet)\end{aligned}$$

in Mod A.

(2) $\Rightarrow$ (1). Since $H^i(\boldsymbol{R}\,\mathrm{Hom}^\bullet(V^\bullet,\ V^\bullet)) = 0$ for $i\ \ 0$, by Proposition 11.7

$$\begin{aligned}A &\cong \mathrm{End}_{D(\mathrm{Mod}\, B^{\mathrm{op}})}(V^\bullet)\\ &\cong \mathrm{Ext}^0(V^\bullet,\ V^\bullet)\\ &\cong H^0(\boldsymbol{R}\,\mathrm{Hom}^\bullet(V^\bullet,\ V^\bullet))\\ &\cong \boldsymbol{R}\,\mathrm{Hom}^\bullet(V^\bullet,\ V^\bullet))\end{aligned}$$

in $D(\mathrm{Mod}\, A)$.

Lemma 25.3. *Let A be a left coherent ring and B a right coherent ring. Then for any* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *the following hold.*

(1) *If* $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, B^{\mathrm{op}}))$, *then we have a ∂-functor*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(-,\, V^\bullet) : D^-(\mathrm{mod}\, A)_{\mathrm{fpd}} \to D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}}.$$

(2) *If A is* $V^\bullet$*-reflexive, then there exists a natural isomorphism*

$$X^\bullet \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\, V^\bullet),\, V^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A)_{\mathrm{fpd}})$.

Proof. Let $X^\bullet \in \mathrm{Ob}(D^-(\mathrm{mod}\, A)_{\mathrm{fpd}})$. Then by Proposition 11.17 we may assume $X^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, A))$. Also, by Lemma 23.4 we have a triangle of the form

$$(\tau_{>n}(X^\bullet),\, \tau_{\geq n}(X^\bullet),\, T^{-n}(X^n),\, \cdot,\, \cdot,\, \cdot).$$

Note that $\tau_{>n}(X^\bullet) = 0$ for $n \gg 0$, and that $X^\bullet = \tau_{>n}(X^\bullet)$ for $n \ll 0$.

(1) For $n \gg 0$, since $\tau_{>n}(X^\bullet) = 0$, $\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet),\, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}})$. Let $n \in \mathbb{Z}$ and assume $\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet),\, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}})$. Note that, since $\boldsymbol{R}\mathrm{Hom}^\bullet(A,\, V^\bullet) \cong \mathrm{Hom}^\bullet(A,\, V^\bullet) \cong V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, B^{\mathrm{op}}))$, $\boldsymbol{R}\mathrm{Hom}^\bullet(T^{-n}(X^n),\, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}})$. Thus, since we have a triangle in $D^-(\mathrm{mod}\, B^{\mathrm{op}})$ of the form

$$(\boldsymbol{R}\mathrm{Hom}^\bullet(T^{-n}(X^n),\, V^\bullet),\, \boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{\geq n}(X^\bullet),\, V^\bullet),\, \boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet),\, V^\bullet),\, \cdot,\, \cdot,\, \cdot),$$

$\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{\geq n}(X^\bullet),\, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}})$. It follows by induction that $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet,\, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}})$.

(2) By Lemma 25.1 there exists a homomorphism of ∂-functors

$$\eta : \mathbf{1}_{D(\mathrm{Mod}\, A)} \to \boldsymbol{R}\mathrm{Hom}^\bullet(-,\, V^\bullet) \circ \boldsymbol{R}\mathrm{Hom}^\bullet(-,\, V^\bullet).$$

Note that for $n \gg 0$, since $\tau_{>n}(X^\bullet) = 0$, $\eta(\tau_{>n}(X^\bullet))$ is an isomorphism. Let $n \in \mathbb{Z}$ and assume that $\eta(\tau_{>n}(X^\bullet))$ is an isomorphism. Since $\eta(A)$ is an isomorphism, it follows that $\eta(T^{-n}(X^n))$ is an isomorphism. Thus by Proposition 6.6 $\eta(\tau_{\geq n}(X^\bullet))$ is an isomorphism. It follows by induction that $\eta(X^\bullet)$ is an isomorphism.

Proposition 25.4. *Let A be a left coherent ring and B a right coherent ring. Let* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *such that* $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, A))$ *and* $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{proj}\, B^{\mathrm{op}}))$. *Assume both A and B are* $V^\bullet$*-reflexive. Then* $\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet)$ *defines a duality between* $D^-(\mathrm{mod}\, A)_{\mathrm{fpd}}$ *and* $D^-(\mathrm{mod}\, B^{\mathrm{op}})_{\mathrm{fpd}}$.

Proof. By Lemma 25.3.

Corollary 25.5. *Let A be a left and right coherent ring. Then* $\boldsymbol{R}\mathrm{Hom}^\bullet(-, A)$ *defines a duality between* $D^-(\mathrm{mod}\, A)_{\mathrm{fpd}}$ *and* $D^-(\mathrm{mod}\, A^{\mathrm{op}})_{\mathrm{fpd}}$.

Lemma 25.6. *For* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *such that* $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A)_{\mathrm{fid}})$. *Then we have a ∂-functor*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) : D^{\mathrm{b}}(\mathrm{Mod}\, A) \to D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}).$$

Proof. Let $X \in \mathrm{Mod}\, A$. Since $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A))$, $\mathrm{Ext}^i(X, V^\bullet) = 0$ for $i \ll 0$ and hence $\boldsymbol{R}\mathrm{Hom}^\bullet(X, V^\bullet) \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}}))$. Also, since $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A)_{\mathrm{fid}})$, $\mathrm{Ext}^i(X, V^\bullet) = 0$ for $i \gg 0$ and $\boldsymbol{R}\mathrm{Hom}^\bullet(X, V^\bullet) \in \mathrm{Ob}(D^-(\mathrm{Mod}\, B^{\mathrm{op}}))$. Thus $\boldsymbol{R}\mathrm{Hom}^\bullet(X, V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$. Next, let $X^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A))$. Then by Lemma 23.4 we have a triangle in $D^{\mathrm{b}}(\mathrm{Mod}\, A)$ of the form

$$(\tau_{>n}(X^\bullet), \tau_{\geq n}(X^\bullet), T^{-n}(X^n), \cdot, \cdot, \cdot).$$

Note that $\tau_{>n}(X^\bullet) = 0$ for $n \gg 0$, and that $X^\bullet = \tau_{>n}(X^\bullet)$ for $n \ll 0$. For $n \gg 0$, since $\tau_{>n}(X^\bullet) = 0$, $\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet), V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$. Let $n \in \mathbb{Z}$ and assume $\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet), V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$. Then, since $\boldsymbol{R}\mathrm{Hom}^\bullet(T^{-n}(X^n), V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$, and since we have a triangle in $D^-(\mathrm{mod}\, B^{\mathrm{op}})$ of the form

$$(\boldsymbol{R}\mathrm{Hom}^\bullet(T^{-n}(X^n), V^\bullet), \boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{\geq n}(X^\bullet), V^\bullet), \boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{>n}(X^\bullet), V^\bullet), \cdot, \cdot, \cdot),$$

$\boldsymbol{R}\mathrm{Hom}^\bullet(\tau_{\geq n}(X^\bullet), V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$. It follows by induction that $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$.

Lemma 25.7. *Let A be left coherent. Let* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *such that* $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$. *Then the following hold.*

(1) *If A is* $V^\bullet$*-reflexive, then there exists a natural isomorphism*

$$X^\bullet \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), V^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D_c^-(\mathrm{Mod}\, A))$.

(2) *If A is $V^\bullet$-reflexive and* $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A)_{\mathrm{fid}})$, *then there exists a natural isomorphism*

$$X^\bullet \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^\bullet(\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet), V^\bullet)$$

for $X^\bullet \in \mathrm{Ob}(D_c(\mathrm{Mod}\, A))$.

(3) *If B is right coherent and* $V^\bullet \in \mathrm{Ob}(D_c(\mathrm{Mod}\, A))$, *then we have a ∂-functor*

$$\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) : D_c(\mathrm{Mod}\, B^{\mathrm{op}}) \to D_c(\mathrm{Mod}\, A).$$

Proof. (1) It follows by Propositions 22.7(1) and 22.8 that

$$\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) \circ \boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) : D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, A)$$

is way-out left. Thus Proposition 23.5(5) applies.

(2) It follows by Proposition 22.8 that

$$\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) \circ \boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) : D(\mathrm{Mod}\, A) \to D(\mathrm{Mod}\, A)$$

is way-out in both directions. Thus Proposition 23.5(5) and then Proposition 23.5(6) apply, successively.

(3) By Proposition 22.8 $\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet) : D(\mathrm{Mod}\, B^{\mathrm{op}}) \to D(\mathrm{Mod}\, A)$ is way-out in both directions. Thus, since $\boldsymbol{R}\mathrm{Hom}^\bullet(B, V^\bullet) \cong V^\bullet \in \mathrm{Ob}(D_c(\mathrm{Mod}\, A))$, Proposition 23.6(5) and then Proposition 23.6(6) apply, successively.

Proposition 25.8. *Let A be a left coherent ring and B a right coherent ring. Let* $V^\bullet \in \mathrm{Ob}(D^b(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *such that* $V^\bullet \in \mathrm{Ob}(D_c^+(\mathrm{Mod}\, A)_{\mathrm{fid}})$ *and* $V^\bullet \in \mathrm{Ob}(D_c^+(\mathrm{Mod}\, B^{\mathrm{op}})_{\mathrm{fid}})$. *Then the following are equivalent.*

(1) $\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet)$ *defines a duality between* $D_c(\mathrm{Mod}\, A)$ *and* $D_c(\mathrm{Mod}\, B^{\mathrm{op}})$.

(2) $\boldsymbol{R}\mathrm{Hom}^\bullet(-, V^\bullet)$ *defines a duality between* $D_c^b(\mathrm{Mod}\, A)$ *and* $D_c^b(\mathrm{Mod}\, B^{\mathrm{op}})$.

(3) *Both A and B are* $V^\bullet$*-reflexive*.

Proof. (1) ⇒ (2). Let $X^\bullet \in \mathrm{Ob}(D_c^b(\mathrm{Mod}\, A))$. Then by Lemma 25.6 $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D^b(\mathrm{Mod}\, B^{\mathrm{op}}))$. Thus $\boldsymbol{R}\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D_c^b(\mathrm{Mod}\, B^{\mathrm{op}}))$. Similarly, $\boldsymbol{R}\mathrm{Hom}^\bullet(M^\bullet, V^\bullet) \in \mathrm{Ob}(D_c^b(\mathrm{Mod}\, A))$ for all $M^\bullet \in \mathrm{Ob}(D_c^b(\mathrm{Mod}\, B^{\mathrm{op}}))$.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). By Lemma 25.7.

Definition 25.3. Let A be a left and right coherent ring. Then a bounded complex $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R A^{\mathrm{op}}))$ is called a dualizing complex if it satisfies the following conditions:

(a) $V^\bullet \in \mathrm{Ob}(D^+_{\mathrm{c}}(\mathrm{Mod}\, A)_{\mathrm{fid}})$;

(b) $V^\bullet \in \mathrm{Ob}(D^+_{\mathrm{c}}(\mathrm{Mod}\, A^{\mathrm{op}})_{\mathrm{fid}})$; and

(c) $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, V^\bullet)$ defines a duality between $D_{\mathrm{c}}(\mathrm{Mod}\, A)$ and $D_{\mathrm{c}}(\mathrm{Mod}\, A^{\mathrm{op}})$.

Lemma 25.9. *Let A be a left coherent ring and B a right coherent ring. Let* $V^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, A \otimes_R B^{\mathrm{op}}))$ *such that* $V^\bullet \in \mathrm{Ob}(D^+(\mathrm{Mod}\, A)_{\mathrm{fid}})$ *and* $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{mod}\, B^{\mathrm{op}}))$. *Then there exists a ∂-functor*

$$\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, V^\bullet) : D^{\mathrm{b}}(\mathrm{mod}\, A) \to D^{\mathrm{b}}(\mathrm{mod}\, B^{\mathrm{op}}).$$

Proof. Let $X^\bullet \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{mod}\, A))$. Then by Proposition 10.15 we may assume $X^\bullet \in \mathrm{Ob}(K^{-,\mathrm{b}}(\mathrm{proj}\, A))$. Thus, since $V^\bullet \in \mathrm{Ob}(K^{\mathrm{b}}(\mathrm{mod}\, B^{\mathrm{op}}))$, $\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \cong \mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{mod}\, B^{\mathrm{op}}))$. Also, by Lemma 25.6 $\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{Mod}\, B^{\mathrm{op}}))$. It follows that $\boldsymbol{R}\,\mathrm{Hom}^\bullet(X^\bullet, V^\bullet) \in \mathrm{Ob}(D^{\mathrm{b}}(\mathrm{mod}\, B^{\mathrm{op}}))$.

Corollary 25.10. *Let A be a left and right coherent ring with* $\mathrm{inj\,dim}\, {}_AA < \infty$. *Then there exists a ∂-functor*

$$\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, A) : D^{\mathrm{b}}(\mathrm{mod}\, A) \to D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}}),$$

which induces a ∂-functor

$$D^{\mathrm{b}}(\mathrm{mod}\, A)/D^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}} \to D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})/D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})_{\mathrm{fpd}}.$$

Proof. By Lemma 25.9 and Corollary 25.5.

Proposition 25.11. *Let A be a left and right coherent ring with* $\mathrm{inj\,dim}\, {}_AA = \mathrm{inj\,dim}\, A_A < \infty$. *Then A itself is a dualizing complex. In particular,* $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, A)$ *induces a duality between* $D^{\mathrm{b}}(\mathrm{mod}\, A)/D^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}}$ *and* $D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})/D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})_{\mathrm{fpd}}$.

Proof. It follows by Proposition 25.8 that A itself is a dualizing complex. Then by Corollary 25.b $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, A)$ defines a duality between $D^{\mathrm{b}}(\mathrm{mod}\, A)$ and $D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})$. Thus by Corollary 25.5 $\boldsymbol{R}\,\mathrm{Hom}^\bullet(-, A)$ induces a duality between $D^{\mathrm{b}}(\mathrm{mod}\, A)/D^{\mathrm{b}}(\mathrm{mod}\, A)_{\mathrm{fpd}}$ and $D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})/D^{\mathrm{b}}(\mathrm{mod}\, A^{\mathrm{op}})_{\mathrm{fpd}}$.

Proposition 25.12. *There exist natural isomorphisms*

$$\boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet, E^\bullet) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet)),$$
$$\boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet \overset{\boldsymbol{L}}{\otimes} X^\bullet, E^\bullet) \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, \boldsymbol{R}\operatorname{Hom}^\bullet(M^\bullet, E^\bullet))$$

for $M^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A^{\mathrm{op}}))$, $X^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A))$ *and* $E^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} R))$.

Proof. By Proposition 24.1.

Proposition 25.13. $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \in \operatorname{Ob}(D(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fid}})$ *for all* $X^\bullet \in \operatorname{Ob}(D(\operatorname{Mod} A)_{\mathrm{fTd}})$ *and* $E^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} R)_{\mathrm{fid}})$.

Proof. By Lemma 20.7 and Proposition 11.12 we may assume $X^\bullet \in \operatorname{Ob}(K^+(\operatorname{Flat} A))$ and $E^\bullet \in \operatorname{Ob}(K^{\mathrm{b}}(\operatorname{Inj} R))$, respectively. Then, for any $M \in \operatorname{Mod} A^{\mathrm{op}}$, since $\operatorname{Hom}^\bullet(M \otimes X^\bullet, E^\bullet) \in \operatorname{Ob}(K^-(\operatorname{Mod} R))$, by Proposition 25.12

$$\begin{aligned}
\operatorname{Ext}^i(M, \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet)) &\cong H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(M, \boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet))) \\
&\cong H^i(\boldsymbol{R}\operatorname{Hom}^\bullet(M \overset{\boldsymbol{L}}{\otimes} X^\bullet, E^\bullet)) \\
&\cong H^i(\operatorname{Hom}^\bullet(M \otimes X^\bullet, E^\bullet)) \\
&= 0
\end{aligned}$$

for $i \gg 0$.

Proposition 25.14. *Let A be left coherent. Then there exists a natural isomorphism*

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Y^\bullet, X^\bullet), E^\bullet)$$

for $X^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A))$, $Y^\bullet \in \operatorname{Ob}(D^-_{\mathrm{c}}(\operatorname{Mod} A))$ *and* $E^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} R)_{\mathrm{fid}})$.

Proof. By Proposition 24.8.

Proposition 25.15. *Let A be left coherent. Then* $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \in \operatorname{Ob}(D^-(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fTd}})$ *for all* $X^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A)_{\mathrm{fid}})$ *and* $E^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} R)_{\mathrm{fid}})$.

Proof. By Proposition 11.12 we may assume $X^\bullet \in \operatorname{Ob}(K^{\mathrm{b}}(\operatorname{Inj} A))$. Also, we may assume $E^\bullet \in \operatorname{Ob}(K^{\mathrm{b}}(\operatorname{Inj} R))$. Note that $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \cong \operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \in \operatorname{Ob}(D^{\mathrm{b}}(\operatorname{Mod} A^{\mathrm{op}}))$. Next, for any $Y \in \operatorname{mod} A$ and $i \gg 0$, $\operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(Y, X^\bullet), E^\bullet) \in \operatorname{Ob}(K^{\mathrm{b}}(\operatorname{Mod} R))$ and hence by Proposition 25.14

$$\begin{aligned}\operatorname{Tor}_i(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet), Y) &\cong H^{-i}(\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \overset{\boldsymbol{L}}{\otimes} Y)\\ &\cong H^{-i}(\boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Y, X^\bullet), E^\bullet))\\ &\cong H^{-i}(\operatorname{Hom}^\bullet(\operatorname{Hom}^\bullet(Y, X^\bullet), E^\bullet))\\ &= 0.\end{aligned}$$

Thus by Lemma 20.7 $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \in \operatorname{Ob}(D^-(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fTd}})$.

Proposition 25.16. *Let A be left coherent. Then there exists a natural isomorphism*

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Y^\bullet, X^\bullet), E^\bullet)$$

for $X^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A)_{\mathrm{fid}})$, $Y^\bullet \in \operatorname{Ob}(D_{\mathrm{c}}(\operatorname{Mod} A))$ *and* $E^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} R)_{\mathrm{fid}})$.

Proof. Let $X^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} A)_{\mathrm{fid}})$ and $E^\bullet \in \operatorname{Ob}(D^+(\operatorname{Mod} R)_{\mathrm{fid}})$. By Lemma 24.7(2) there exists a natural isomorphism

$$\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \overset{\boldsymbol{L}}{\otimes} Y^\bullet \xrightarrow{\sim} \boldsymbol{R}\operatorname{Hom}^\bullet(\boldsymbol{R}\operatorname{Hom}^\bullet(Y^\bullet, X^\bullet), E^\bullet)$$

for $Y^\bullet \in \operatorname{Ob}(D^-(\operatorname{mod} A))$. By Proposition 25.15 $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \in \operatorname{Ob}(D^-(\operatorname{Mod} A^{\mathrm{op}})_{\mathrm{fTd}})$ and hence by Proposition 22.17 $\boldsymbol{R}\operatorname{Hom}^\bullet(X^\bullet, E^\bullet) \overset{\boldsymbol{L}}{\otimes} -$ is way-out in both directions. Also, it follows by Proposition 22.8 that $\boldsymbol{R}\operatorname{Hom}^\bullet(-, E^\bullet) \circ \boldsymbol{R}\operatorname{Hom}^\bullet(-, X^\bullet)$ is way-out in both directions. Thus, the assertion follows by Proposition 23.5(6).