Modules of the Highest Homological Dimension over a Gorenstein Ring

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Dedicated to Professor Kent R. Fuller on his 60th birthday

Abstract. We will study modules of the highest injective, projective and flat dimension over a Goresntein ring. Let R be a Gorenstein ring of self-injective dimension n and $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ a minimal injective resolution. Then it is shown in [F-I] that the flat dimension and projective dimension of E_n is n, the highest dimension. In this note, we shall prove that if M is a left R-module of injective dimension n, then the last injective term $E^n(M)$ in a minimal injective resolution of M has projective and flat dimension n, and any indecomposable summand of $E^n(M)$ embeds in E_n. As a consequence, we obtain that if R is Auslander-Gorenstein, then $E^n(M)$ has essential socle.

1. Introduction

A Noether ring R is called **Gorenstein** if R has left and right finite self-injective dimensions. Further, a Noether ring R is called **Auslander-Gorenstein** if R is Gorenstein and in a minimal injective resolution $0 \to {}_RR \to E_0 \to E_1 \to \cdots$, each E_i has flat dimension at most i. This concept was introduced by Auslander as

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a non-commutative version of the Gorenstein condition for commutative rings, studied by Bass [Ba]. In the non-commutative case, we can also see the ubiquity of Auslander-Gorenstein rings in several articles, for example, [B-G], [G-L], [Le], [L-S], [S-Z].

We showed in [Iw1] that for a Gorenstein ring of self-injective dimension n, finiteness of the injective, projective and flat dimensions of a module are all equivalent, and all of these dimensions are at most n. This fact motivated our interest in modules with the highest injective, projective or flat dimension. Let R be a Gorenstein ring of self-injective dimension n and $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ a minimal injective resolution. Then it is shown in [F-I] that any direct summand of E_n has the highest projective and flat dimension n. In this note, we will study the relationship between more general modules of projective (or flat) dimension n and the module E_n.

Throughout this note, $\mathrm{id}(M)$, $\mathrm{pd}(M)$ and $\mathrm{fd}(M)$ stand for the injective, projective and flat dimension of a module M, respectively. Further, $0 \to M \to E^0(M) \to \cdots \to E^n(M) \to \cdots$ is a minimal injective resolution of M.

The results obtained in this note are the following.

Theorem 1. *Let R be a Gorenstein ring of self-injective dimension n and $0 \to {}_RR \to E_0 \to \cdots \to E_n \to 0$ a minimal injective resolution. If a left R-module M has injective dimension n, then any indecomposable direct summand E of $E^n(M)$ is isomorphic to a summand in E_n. As a consequence E has projective and flat dimension n.*

As a byproduct, [Mi2, Corollary 1.3] and [I-S2, Theorem 6] yield a generalization of [I-S2, Theorem 6] for Auslander-Gorenstein rings. Hoshino showed that an injective indecomposable module of flat dimension i over an Auslander-Gorenstein ring appears in i-th injective term of a minimal injective resolution of the ring ([Ho, Theorem 6.3]). Miyachi showed that any injective indecomposable module over a Gorenstein ring appears in some injective term of a minimal injective resolution of the ring ([Mi1, Corollary 4.7]).

Theorem 2. *If R is an Auslander-Gorenstein ring of self-injective dimension n, then any injective indecomposable left R-module of flat dimension n is isomorphic to a direct summand of E_n and is* *of the form $E(S)$ for a simple left module S. Thus if a left R-module M has injective dimension n, $E^n(M)$ has essential socle.*

The final result generalizes [I-S1, Theorem; I-S2, Theorem 2]. It appears interesting to study the distribution of injective indecomposables along the terms of a minimal injective resolution of a Gorenstein ring.

Proposition 3. *Let R be a Noether ring and $0 \to {}_RR \to E_0 \to \cdots \to E_n \to \cdots$ a minimal injective resolution of ${}_RR$.*

(1) If M is a left R-module with $0 < i = id(M) < \infty$, then E_0 and $E^i(M)$ have no isomorphic direct summands in common.

(2) If R has left self-injective dimension $n \geq 1$, then E_0 and E_n have no isomorphic direct summands in common.

2. The Proofs

Proof of Theorem 1.

By [Iw1, Theorem 2], M has projective dimension at most n. Thus let $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M and consider an injective resolution of each P_i $(0 \leq i \leq n)$

$$0 \to P_i \to E^0(P_i) \to E^1(P_i) \to \ \cdots \ \to E^n(P_i) \to 0.$$

Then $E^j(P_i)$ for each j $(0 \leq j \leq n)$ is a direct summand of a direct sum of copies of E_j. Hence by [Mi2, Corollary 1.3], M has an injective resolution of the following form

$$\begin{aligned} 0 \ \to \ M \ \to \ Q \longrightarrow \bigoplus_{i=0}^{n-1} E^{i+1}(P_i) \longrightarrow \bigoplus_{i=0}^{n-2} E^{i+2}(P_i) \longrightarrow \ \cdots \\ \longrightarrow E^{n-1}(P_0) \oplus E^n(P_1) \longrightarrow E^n(P_0) \ \to \ 0. \end{aligned}$$

Here Q is a direct summand of $\bigoplus_{i=0}^{n} E^i(P_i)$. Then $E^n(M)$ is a direct summand of $E^n(P_0)$, and so a direct summand of direct sum of copies of E_n. Since any indecomposable summand E of $E^n(M)$ is uniform, E embeds in E_n. $\square$

Proof of Theorem 2.

Let E be an injective indecomposable left module of flat dimension n. By [Mi1, Corollary 4.7], E is isomorphic to a direct summand in E_n. Since $\mathrm{Soc}(E_n)$ is essential in E_n ([I-S 2, Theorem 6]), E is of the form $\mathrm{E}(S)$ for some simple module S.

By Theorem 1 and [F-I, Proposition 1.1], any direct summand of $E^n(M)$ has flat dimension n and so has essential socle. That is, the socle of $E^n(M)$ is essential. □

Proof of Proposition 3.

(1) Let U be any nonzero submodule of E_0 and $V = U \cap R \neq 0$. Then from the exact sequence

$$0 \to V \to R \to R/V \to 0,$$

we have an exact sequence

$$\mathrm{Ext}^i_R(R,\ M) \longrightarrow \mathrm{Ext}^i_R(V,\ M) \longrightarrow \mathrm{Ext}^{i+1}_R(R/V,\ M).$$

Here $\mathrm{Ext}^i_R(R,\ M) = 0$ from $i > 0$ and $\mathrm{Ext}^{i+1}_R(R/V,\ M) = 0$ from $\mathrm{id}(M) = i$. Hence we obtain $\mathrm{Ext}^i_R(V,\ M) = 0$ and thus we see that V is not monomorphic to $E^i(M)$.

(2) is obvious from (1). □

3. Examples

Let us conclude this note with a few examples. In the following examples, if R is a path algebra given by a quiver $\mathcal{Q}$ with set $\mathcal{Q}_0$ of vertices and $i \in \mathcal{Q}_0$, then $S(i)$ denotes the simple R-module corresponding to the vertex i and $E(i)$ its injective hull.

(1) Theorem 1 and Proposition 3 prompt us to raise the following question: *Let R be a Gorenstein ring of self-injective dimension n and E an injective indecomposable R-module of projective dimension n. Then does there exist an R-module M of injective dimension n such that E embeds in $E^n(M)$?* It's easy to see that the question is affirmative if R is Auslander-Gorenstein. However, the answer is negative for Gorenstein rings. For example, let R be

a finite dimensional algebra over any field given by the following quiver

$$\begin{array}{ccccccc} 1 & & & & & & \\ \alpha \searrow & & & & & & \\ & & 3 & \xrightarrow{\gamma} & 4 & \underset{\varepsilon}{\overset{\delta}{\rightleftarrows}} & 5 \\ \beta \nearrow & & & & & & \\ 2 & & & & & & \end{array}$$

with the relations $\gamma\alpha = \gamma\beta = \varepsilon\delta = \delta\varepsilon = 0$. Then R is a Gorenstein ring of self-injective dimension 2 and has infinite global dimension. $E(3)$ has projective dimension 2 but never appears in $\mathrm{E}^2(M)$ for any R-module M of injective dimension 2. Also we can see from this observation that an injective indecomposable module with the highest projective dimension does not necessarily embed in the last term of a minimal injective resolution of a Gorenstein ring.

Moreover, $E(1)$ and $E(2)$ are both direct summands of the last injective term E_2 in a minimal injective resolution $0 \to {}_RR \to E_0 \to E_1 \to E_2 \to 0$ but $\mathrm{Ext}^1_R(E(i),\ R) \neq 0$ $(i = 1, 2)$. Hence $E(1)$ and $E(2)$ are not holonomic. Here a finitely generated module X over a Gorenstein ring R of self-injective dimension n is called **holonomic** if $\mathrm{Ext}^i_R(X,\ R) = 0$ for all $i \neq n$.

Finally we can see in this example that all injective terms $E_0,\ E_1$ and E_2 have the highest projective and flat dimensions.

(2) In [Iw2], it is proved that any holonomic module over an Auslander-Gorenstein ring has finite composition length and embeds in a direct sum of finitely many copies of the last injective term in a minimal injective resolution of the ring. However a submodule of finite composition length in the last injective term is not necessarily holonomic.

For example, let R be a finite dimensional algebra over any field given by the following quiver

$$\begin{array}{ccccc} & & 2 & & \\ \alpha \nearrow & & & & \searrow \gamma \\ 1 & \overset{\mu}{\leftarrow} & 5 & \overset{\lambda}{\leftarrow} & 4 \\ \beta \searrow & & & & \nearrow \delta \\ & & 3 & & \end{array}$$

with the relations $\mu\lambda = \alpha\mu = \beta\mu = 0$ and $\gamma\alpha = \delta\beta$. Then R is Auslander-Gorenstein of self-injective dimension 4.

Consider the left R-module M of dimension vector $(0, 1, 1, 1, 0)$, then M is a submodule of the last injective term of a minimal injective resolution of ${}_RR$ but not holonomic. For, we can see $\mathrm{Ext}^1_R(M,\ R) \neq 0$, that is, the grade of M is one.

(3) We can see that if R is a Gorenstein ring of self-injective dimension n and S is a simple submodule of the last injective term E_n in a minimal injective resolution of ${}_RR$, then $\mathrm{pd}(S) = \mathrm{fd}(S) = n$ or ∞. Conversely, if S is a simple R-module of the highest projective dimension n, S appears in the socle of E_n. There is an example of a Gorenstein ring R with a simple module of infinite projective and flat dimension not appearing in E_n.

Let R be a finite dimensional algebra over any field given by the following quiver

$$\begin{array}{ccc} 1 & \overset{\delta}{\longleftarrow} & 3 \\ \alpha \searrow & & \nearrow \gamma \\ & 2 & \\ & \circlearrowright \beta & \end{array}$$

with the relations $\alpha\delta = \gamma\alpha = \beta^2 = 0$. Then R is Auslander-Gorenstein of self-injective dimension 3. We can see

$$\mathrm{pd}(S(1)) = 2, \quad \mathrm{pd}(S(2)) = \infty, \quad \mathrm{pd}(S(3)) = 3$$

and

$$E_0 = E(1)^{(4)}, \quad E_1 = E(2)^{(2)}, \quad E_2 = E(1), \quad E_3 = E(3).$$

Here, $M^{(t)}$ stands for a direct sum of t copies of a module M.

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