t-STRUCTURES, TORSION THEORIES AND DG ALGEBRAS

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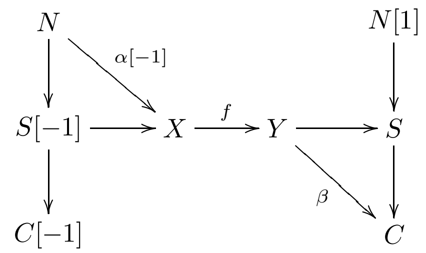
In this note, for a ring A $\mathsf{Mod}\, A$ (resp., $\mathsf{mod}\, A$) is the category of right A-modules (resp., finitely generated right A-modules), and $\mathsf{Proj}\, A$ (resp., $\mathsf{proj}\, A$) the category of projective right A-modules (resp., finitely generated projective right A-modules).

1. t-STRUCTURES

We recall the notion of t-structures which was introduced by Beilinson, Bernstein and Deligne. In this section, $\mathcal{T}$ is a triangulated category, $\mathcal{C}$ is a full subcategory of $\mathcal{T}$ satisfying

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{C}[i]) = 0 \quad (i < 0).$$

Proposition 1.1. *For a morphism $f : X \to Y$ in $\mathcal{C}$, suppose that there are $N, C \in \mathcal{C}$ such that*



where all vertical and horizontal sequences are distinguished triangles. Then we have $\ker f = \alpha[-1]$, $\mathrm{Cok}\, f = \beta$ *in $\mathcal{C}$.*

Definition 1.2. *A morphism $f : X \to Y$ in $\mathcal{C}$ is called $\mathcal{C}$-admissible if there exist $N, C \in \mathcal{C}$ satisfying Proposition 1.1. A sequence $X \to Y \to Z$ in $\mathcal{C}$ is called an admissible short exact sequence if $X \to Y \to Z \to X[1]$ is a distinguish triangle for some $Z \to X[1]$.*

Proposition 1.3. *Suppose that $\mathcal{C}$ is stable under finite coproducts. Then the following are equivalent.*

1. *$\mathcal{C}$ is abelian, and all short exact sequences are admissible.*
2. *All morphisms in $\mathcal{C}$ are $\mathcal{C}$-admissible.*

Definition 1.4. *A full subcategory $\mathcal{C}$ of $\mathcal{T}$ is called an admissible abelian category if $\mathcal{C}$ satisfy the equivalent conditions in Proposition 1.3.*

Definition 1.5. *Let $\mathcal{T}$ be a triangulated category. For full subcategories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called a t-structure on $\mathcal{T}$ provided that*

(i) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$;
(ii) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ *and* $\mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1}$;

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(iii) *for any* $X \in \mathcal{T}$, *there exists a distinguished triangle*

$$X' \to X \to X'' \to$$

with $X' \in \mathcal{T}^{\le 0}$ *and* $X'' \in \mathcal{T}^{\ge 1}$,

where $\mathcal{T}^{\le n} = \mathcal{T}^{\le 0}[-n]$ *and* $\mathcal{T}^{\ge n} = \mathcal{T}^{\ge 0}[-n]$.

The core of this t-structure is $\mathcal{C} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$.

Proposition 1.6. *For* $n \in \mathbb{Z}$, *the following hold.*

1. *The inclusion* $\mathcal{T}^{\le n} \to \mathcal{T}$ *has a right adjoint* $\sigma_{\le n} : \mathcal{T} \to \mathcal{T}^{\le n}$.
2. *The inclusion* $\mathcal{T}^{\ge n} \to \mathcal{T}$ *has a left adjoint* $\sigma_{\ge n} : \mathcal{T} \to \mathcal{T}^{\ge n}$.
3. *For any* $X \in \mathcal{T}$, *there exists a unique* $d \in \mathrm{Hom}_{\mathcal{T}}(\sigma_{\ge 1}X, \sigma_{\le 0}X[1])$ *such that*

$$\sigma_{\le 0}X \to X \to \sigma_{\ge 1}X \xrightarrow{d} \sigma_{\le 0}X[1]$$

is a distinguished triangle.

4. *Let* $A \to X \to B \to A[1]$ *be a distinguished triangle with* $A \in \mathcal{T}^{\le 0}$, $B \in \mathcal{T}^{\ge 1}$. *Then this triangle is isomorphic to* $\sigma_{\le 0}X \to X \to \sigma_{\ge 1}X \xrightarrow{d} \sigma_{\le 0}X[1]$.

Remark 1.7. *For* $X \in \mathcal{T}$, *the following hold.*

1. $\sigma_{\ge n}X = O$ *iff* $X \in \mathcal{T}^{\le n-1}$.
2. $\sigma_{\le n}X = O$ *iff* $X \in \mathcal{T}^{\ge n+1}$.

Proposition 1.8. *For* $a \le b$, $X \in \mathcal{T}$, *there is an isomorphism* $\sigma_{\ge a}\sigma_{\le b}X \xrightarrow{\sim} \sigma_{\le b}\sigma_{\ge a}X$ *such that*

$$\begin{array}{ccccc}
\sigma_{\ge a}\sigma_{\le b}X & & \xrightarrow{\quad\sim\quad} & & \sigma_{\le b}\sigma_{\ge a}X \\
\uparrow & & & & \downarrow \\
\sigma_{\le b}X & \longrightarrow & X & \longrightarrow & \sigma_{\ge a}X
\end{array}$$

is commutative.

Theorem 1.9. *The core* $\mathcal{C} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$ *is an admissible abelian category which is stable under extensions, and* $\mathrm{H}^0 = \sigma_{\ge 0}\sigma_{\le 0} : \mathcal{T} \to \mathcal{C}$ *is a cohomological functor.*

Definition 1.10. *A t-structure* $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ *on* $\mathcal{T}$ *is called non-degenerate provided that* $\bigcap_{n\in\mathbb{Z}}\mathcal{T}^{\le n} = \bigcap_{n\in\mathbb{Z}}\mathcal{T}^{\ge n} = \{0\}$.

Proposition 1.11. *Let* $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ *be a non-degenerate t-structure. For* $X \in \mathcal{T}$, *the following hold.*

1. $\mathrm{H}^i X = O$ *for any* n *iff* $X = O$.
2. $\mathrm{H}^i X = O$ *for any* $i > n$ *(resp.,* $i < n$*) iff* $X \in \mathcal{T}^{\le n}$ *(resp.,* $X \in \mathcal{T}^{\ge n}$*).*

Here $\mathrm{H}^i X = \mathrm{H}^0(X[i])$.

2. t-structures Induced by Compact Objects

A triangulated category $\mathcal{T}$ is said to contain coproducts if coproducts of objects indexed by any set exist in $\mathcal{T}$. An object C of $\mathcal{T}$ is called compact if $\mathrm{Hom}_{\mathcal{T}}(C, -)$ commutes with coproducts. Furthermore, a collection $\mathcal{S}$ of compact objects of $\mathcal{T}$ is called a generating set provided that $X = 0$ whenever $\mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0$, and that $\mathcal{S}$ is stable under suspensions. In this case, $\mathcal{T}$ is called compactly generated (see [Ne] for details). For an object $C \in \mathcal{T}$ and an integer n, we denote by $\mathcal{T}^{\ge n}(C)$ (resp.,

$\mathcal{T}^{\leq n}(C)$) the full subcategory of $\mathcal{T}$ consisting of $X \in \mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(C, X[i]) = 0$ for $i < n$ (resp., $i > n$), and set $\mathcal{T}^0(C) = \mathcal{T}^{\leq 0}(C) \cap \mathcal{T}^{\geq 0}(C)$.

For an abelian category $\mathcal{A}$, we denote by $\mathsf{C}(\mathcal{A})$ the category of complexes of $\mathcal{A}$, and denote by $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes of $\mathcal{A}$ (resp., complexes of $\mathcal{A}$ with bounded below homologies, complexes of $\mathcal{A}$ with bounded above homologies, complexes of $\mathcal{A}$ with bounded homologies). For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes of $\mathcal{B}$ (resp., bounded above complexes of $\mathcal{B}$, bounded complexes of $\mathcal{B}$) (see [RD] for details).

Proposition 2.1. *Let $\mathcal{T}$ be a triangulated category which contains coproducts, C a compact object satisfying $\mathrm{Hom}_{\mathcal{T}}(C, C[n]) = 0$ for $n > 0$. Then for any $r \in \mathbb{Z}$ and any object $X \in \mathcal{T}$, there exist an object $X^{\geq r} \in \mathcal{T}^{\geq r}(C)$ and a morphism $\alpha^{\geq r} : X \to X^{\geq r}$ in $\mathcal{T}$ such that*

(i) *for any $i \geq r$, $\mathrm{Hom}_{\mathcal{T}}(C, \alpha^{\geq r}[i])$ is an isomorphism,*
(ii) *for every object $Y \in \mathcal{T}^{\geq r}(C)$, $\mathrm{Hom}_{\mathcal{T}}(\alpha^{\geq r}, Y)$ is an isomorphism.*

Theorem 2.2. *Let $\mathcal{T}$ be a triangulated category which contains coproducts, C a compact object satisfying $\mathrm{Hom}_{\mathcal{T}}(C, C[n]) = 0$ for $n > 0$, and $B = \mathrm{End}_{\mathcal{T}}(C)$. If $\{C[i] \mid i \in \mathbb{Z}\}$ is a generating set, then the following hold.*

(1) *$(\mathcal{T}^{\leq 0}(C), \mathcal{T}^{\geq 0}(C))$ is a non-degenerate t-structure on $\mathcal{T}$.*
(2) *$\mathcal{T}^0(C)$ is admissible abelian.*
(3) *The functor*
$$\mathrm{Hom}_{\mathcal{T}}(C, -) : \mathcal{T}^0(C) \to \mathsf{Mod}\, B$$
is an equivalence.

3. Torsion Theories for Abelian Categories

Throughout this section, we fix the following notation. Let $\mathcal{A}$ be an abelian category satisfying the condition Ab4 (i.e. direct sums of exact sequences are exact), and let $d_P^{-1} : P^{-1} \to P^0$ be a morphism in $\mathcal{A}$ with the P^i being small projective objects of $\mathcal{A}$, and denote by $P^{\boldsymbol{\cdot}}$ the mapping cone of d_P^{-1}. We set $\mathcal{C}(P^{\boldsymbol{\cdot}}) = \mathsf{D}(\mathcal{A})^0(P^{\boldsymbol{\cdot}})$, $B = \mathrm{End}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}})$, and define a pair of full subcategories of $\mathcal{A}$
$$\mathcal{X}(P^{\boldsymbol{\cdot}}) = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}}, X[1]) = 0\},$$
$$\mathcal{Y}(P^{\boldsymbol{\cdot}}) = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}}, X) = 0\}.$$
For any $X \in \mathcal{A}$, we define a subobject of X
$$\tau(X) = \sum\nolimits_{f \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(P^{\boldsymbol{\cdot}}), X)} \mathrm{Im}\, f$$
and an exact sequence in $\mathcal{A}$
$$(e_X) : 0 \to \tau(X) \xrightarrow{j_X} X \to \pi(X) \to 0.$$

Remark 3.1. *It is easy to see that $P^{\boldsymbol{\cdot}}$ is a compact object of $\mathsf{D}(\mathcal{A})$, and we have $\mathcal{X}(P^{\boldsymbol{\cdot}}) = \mathsf{D}(\mathcal{A})^{\leq 0}(P^{\boldsymbol{\cdot}}) \cap \mathcal{A}$ and $\mathcal{Y}(P^{\boldsymbol{\cdot}}) = \mathsf{D}(\mathcal{A})^{\geq 1}(P^{\boldsymbol{\cdot}}) \cap \mathcal{A}$.*

Lemma 3.2. *For any $X^{\boldsymbol{\cdot}} \in \mathsf{D}(\mathcal{A})$ and $n \in \mathbb{Z}$, we have a functorial exact sequence*
$$0 \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}}, \mathrm{H}^{n-1}(X^{\boldsymbol{\cdot}})[1]) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}}, X^{\boldsymbol{\cdot}}[n]) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\boldsymbol{\cdot}}, \mathrm{H}^n(X^{\boldsymbol{\cdot}})) \to 0.$$
Moreover, the above short exact sequence commutes with coproducts.

Definition 3.3. *A pair $(\mathcal{X},\mathcal{Y})$ of full subcategories $\mathcal{X},\mathcal{Y}$ in an abelian category $\mathcal{A}$ is called a torsion theory for $\mathcal{A}$ provided that the following conditions are satisfied (see e.g.* [Di] *for details):*

(i) $\mathcal{X}\cap\mathcal{Y}=\{0\}$*;*
(ii) $\mathcal{X}$ *is closed under factor objects;*
(iii) $\mathcal{Y}$ *is closed under subobjects;*
(iv) *for any object X of $\mathcal{A}$, there exists an exact sequence* $0\to X'\to X\to X''\to 0$ *in $\mathcal{A}$ with $X'\in\mathcal{X}$ and $X''\in\mathcal{Y}$.*

Remark 3.4. *Let $\mathcal{A}$ be an abelian category and $(\mathcal{X},\mathcal{Y})$ a torsion theory for $\mathcal{A}$. Then for any $Z\in\mathcal{A}$, the following hold.*

(1) *$Z\in\mathcal{X}$ if and only if* $\mathrm{Hom}_{\mathcal{A}}(Z,\mathcal{Y})=0$.
(2) *$Z\in\mathcal{Y}$ if and only if* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},Z)=0$.

Theorem 3.5. *The following are equivalent for a complex $P^{\cdot}:P^{-1}\xrightarrow{d_P^{-1}}P^0$ with the P^i being small projective objects of $\mathcal{A}$.*

(1) *$\{P^{\cdot}[i]\mid i\in\mathbb{Z}\}$ is a generating set for $\mathsf{D}(\mathcal{A})$ and $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot},P^{\cdot}[i])=0$ for all $i>0$.*
(2) *$\mathcal{X}(P^{\cdot})\cap\mathcal{Y}(P^{\cdot})=\{0\}$ and $\mathrm{H}^0(P^{\cdot})\in\mathcal{X}(P^{\cdot})$.*
(3) *$\mathcal{X}(P^{\cdot})\cap\mathcal{Y}(P^{\cdot})=\{0\}$ and $\tau(X)\in\mathcal{X}(P^{\cdot})$, $\pi(X)\in\mathcal{Y}(P^{\cdot})$ for all $X\in\mathcal{A}$.*
(4) *$(\mathcal{X}(P^{\cdot}),\mathcal{Y}(P^{\cdot}))$ is a torsion theory for $\mathcal{A}$.*

Lemma 3.6. *Assume $\mathcal{X}(P^{\cdot})\cap\mathcal{Y}(P^{\cdot})=\{0\}$. Then for any $X^{\cdot}\in\mathsf{D}(\mathcal{A})$, the following are equivalent.*

(1) *$X^{\cdot}\in\mathcal{C}(P^{\cdot})$.*
(2) *$\mathrm{H}^n(X^{\cdot})=0$ for $n>0$ and $n<-1$, $\mathrm{H}^0(X^{\cdot})\in\mathcal{X}(P^{\cdot})$ and $\mathrm{H}^{-1}(X^{\cdot})\in\mathcal{Y}(P^{\cdot})$.*

Remark 3.7. *Let $\mathcal{A}$ be an abelian category and $\mathcal{X},\mathcal{Y}$ full subcategories of $\mathcal{A}$. Then the pair $(\mathcal{X},\mathcal{Y})$ is a torsion theory for $\mathcal{A}$ if and only if the following two conditions are satisfied:*

(i) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X},\mathcal{Y})=0$*;*
(ii) *for any object X in $\mathcal{A}$, there exists an exact sequence* $0\to X'\to X\to X''\to 0$ *in $\mathcal{A}$ with $X'\in\mathcal{X}$ and $X''\in\mathcal{Y}$.*

Theorem 3.8. *Let $P^{\cdot}$ be a complex $P^{-1}\xrightarrow{d_P^{-1}}P^0$ with the P^i being small projective objects of $\mathcal{A}$. Assume $\mathcal{X}(P^{\cdot})\cap\mathcal{Y}(P^{\cdot})=\{0\}$ and $\mathrm{H}^0(P^{\cdot})\in\mathcal{X}(P^{\cdot})$. Then the following hold.*

(1) *$\mathcal{C}(P^{\cdot})$ is admissible abelian.*
(2) *The functor*
$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot},-):\mathcal{C}(P^{\cdot})\to\mathsf{Mod}\,B$$
is an equivalence.
(3) *$(\mathcal{Y}(P^{\cdot})[1],\mathcal{X}(P^{\cdot}))$ is a torsion theory for $\mathcal{C}(P^{\cdot})$.*

Proposition 3.9. *Assume $P^{\cdot}$ satisfies the conditions*

(i) *$\{P^{\cdot}[i]\mid i\in\mathbb{Z}\}$ is a generating set for $\mathsf{D}(\mathcal{A})$,*
(ii) *$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(P^{\cdot},P^{\cdot}[i])=0$ for $i\neq 0$.*

If $\mathcal{A}$ has either enough projectives or enough injectives, then we have an equivalence of triangulated categories
$$\mathsf{D}^{\mathrm{b}}(\mathcal{A})\cong\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B).$$

Example 3.10 (cf. [HK]). *Let A be a finite dimensional algebra over a field k given by a quiver*

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha} & 2 \\ {\scriptstyle\delta}\big\uparrow & & \big\downarrow{\scriptstyle\beta} \\ 4 & \xleftarrow[\gamma]{} & 3 \end{array}$$

with relations $\beta\alpha = \gamma\beta = \delta\gamma = \alpha\delta = 0$. For each vertex i, we denote by $S(i), P(i)$ the corresponding simple and indecomposable projective left A-modules, respectively. Define a complex $P^\bullet$ as the mapping cone of the homomorphism

$$d_P^{-1} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & 0 & g & 0 \end{bmatrix} : P(2)^2 \oplus P(4)^2 \to P(1) \oplus P(3),$$

where f and g denote the right multiplications of α and γ, respectively. Then $P^\bullet$ is not a tilting complex. However, $P^\bullet$ satisfies the assumption of Theorem 3.8 and hence we have an equivalence of abelian categories

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(P^\bullet, -) : \mathcal{C}(P^\bullet) \to \mathsf{Mod}\,B,$$

where $B = \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\,A)}(P^\bullet)$ is a finite dimensional k-algebra given by a quiver

$$1 \leftarrow 2 \qquad 3 \leftarrow 4.$$

There exist exact sequences in $\mathcal{C}(P^\bullet)$ of the form

$$0 \to S(1) \to S(2)[1] \to P(1)[1] \to 0, \quad 0 \to S(3) \to S(4)[1] \to P(3)[1] \to 0,$$

and these objects and morphisms generate $\mathcal{C}(P^\bullet)$.

In the rest of this section, we deal with the case where R is a commutative artin ring, I is an injective envelope of an R-module $R/\mathrm{rad}(R)$ and A is a finitely generated R-module. We denote by $\mathsf{mod}\,A$ the full abelian subcategory of $\mathsf{Mod}\,A$ consisting of finitely generated modules. $P^\bullet$ is also a complex $P^{-1} \xrightarrow{d_P^{-1}} P^0$ with the P^i being finitely generated projective A-modules. Note that $\mathrm{H}^n(P^\bullet), \mathrm{H}^n(\nu(P^\bullet)) \in \mathsf{mod}\,A$ for all $n \in \mathbb{Z}$. We set

$$\mathcal{X}_c(P^\bullet) = \mathcal{X}(P^\bullet) \cap \mathsf{mod}\,A \quad \text{and} \quad \mathcal{Y}_c(P^\bullet) = \mathcal{Y}(P^\bullet) \cap \mathsf{mod}\,A.$$

Proposition 3.11. *For any tilting complexes $P_1^\bullet : P_1^{-1} \to P_1^0$, $P_2^\bullet : P_2^{-1} \to P_2^0$ for A of term length two, the following are equivalent.*

(1) $(\mathcal{X}_c(P_1^\bullet), \mathcal{Y}_c(P_1^\bullet)) = (\mathcal{X}_c(P_2^\bullet), \mathcal{Y}_c(P_2^\bullet))$.
(2) $\mathsf{add}(P_1^\bullet) = \mathsf{add}(P_2^\bullet)$ *in* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.

Proposition 3.12. *The following are equivalent for a complex $P^{-1} \to P^0 \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$*

(1) $P^\bullet$ *is a tilting complex.*
(2) $\mathcal{X}_c(P^\bullet) \cap \mathcal{Y}_c(P^\bullet) = \{0\}$, $\mathrm{H}^0(P^\bullet) \in \mathcal{X}_c(P^\bullet)$ *and* $\mathrm{H}^{-1}(P^\bullet) \in \mathcal{Y}_c(P^\bullet)$.
(3) $(\mathcal{X}_c(P^\bullet), \mathcal{Y}_c(P^\bullet))$ *is a torsion theory for* $\mathsf{mod}\,A$ *and* $\mathrm{H}^{-1}(P^\bullet) \in \mathcal{Y}_c(P^\bullet)$.
(4) $(\mathcal{X}_c(P^\bullet), \mathcal{Y}_c(P^\bullet))$ *is a torsion theory for* $\mathsf{mod}\,A$ *and* $\mathcal{X}_c(P^\bullet)$ *is stable under* $DA \otimes_A -$.
(5) $(\mathcal{X}_c(P^\bullet), \mathcal{Y}_c(P^\bullet))$ *is a torsion theory for* $\mathsf{mod}\,A$ *and* $\mathcal{Y}_c(P^\bullet)$ *is stable under* $\mathrm{Hom}_A(DA, -)$.

Definition 3.13. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then an object $X \in \mathcal{C}$ is called* Ext*-projective (resp.,* Ext*-injective) if $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, X) = 0$).*

Proposition 3.14. *Assume $P^\bullet$ is a tilting complex. Then the following hold.*

(1) $\mathrm{H}^0(P^\bullet) \in \mathcal{X}_c(P^\bullet)$ *is* Ext*-projective and generates $\mathcal{X}_c(P^\bullet)$.*
(2) $\mathrm{H}^{-1}(\nu(P^\bullet)) \in \mathcal{Y}_c(P^\bullet)$ *is* Ext*-injective and cogenerates $\mathcal{Y}_c(P^\bullet)$.*

Theorem 3.15. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory for $\mathsf{mod}\, A$ such that $\mathcal{X}$ contains an* Ext*-projective module X which generates $\mathcal{X}$, $\mathcal{Y}$ contains an* Ext*-injective module Y which cogenerates $\mathcal{Y}$, and $\mathcal{X}$ is stable under $DA \otimes_A -$. Let $M^\bullet_X$ be a minimal projective presentation of X and $N^\bullet_Y$ a minimal injective presentation of Y. Then*

$$P^\bullet = M^\bullet_X \oplus \mathrm{Hom}^\bullet_A(DA, N^\bullet_Y)[1]$$

is a tilting complex such that $\mathcal{X} = \mathcal{X}_c(P^\bullet)$ and $\mathcal{Y} = \mathcal{Y}_c(P^\bullet)$.

Remark 3.16. *Let*

$$\mathfrak{S} = \{P^\bullet : P^{-1} \to P^0 \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \mid P^\bullet \text{ is a tilting complex for } A\}$$

on which we define the equivalence relation $P^\bullet_1 \sim P^\bullet_2$ provided $\mathsf{add}\, P^\bullet_1 = \mathsf{add}\, P^\bullet_2$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$, and let $\mathfrak{T}$ be the collection of torsion theories $(\mathcal{X}, \mathcal{Y})$ for $\mathsf{mod}\, A$ such that $\mathcal{X}$ contains an Ext*-projective module X which generates $\mathcal{X}$, $\mathcal{Y}$ contains an* Ext*-injective module Y which cogenerates $\mathcal{Y}$, and $\mathcal{X}$ is stable under $DA \otimes_A -$. Set*

$$\Phi(P^\bullet) = ((\mathcal{X}_c(P^\bullet), \mathcal{Y}_c(P^\bullet)) \text{ for } P^\bullet \in \mathfrak{S},$$
$$\Psi((\mathcal{X}, \mathcal{Y})) = M^\bullet_X \oplus \mathrm{Hom}^\bullet_A(DA, N^\bullet_Y)[1] \text{ for } (\mathcal{X}, \mathcal{Y}) \in \mathfrak{T}.$$

Then, according to Propositions 3.11, 3.12, 3.14 and Theorem 3.15, Φ and Ψ induce a one to one correspondence between $\mathfrak{S}/\sim$ and $\mathfrak{T}$.

4. Perverse t-structures Induced by Torsion Theories

We recall the notion of perverse t-structures which was introduced by [BBD] and was translated into the language of torsion theories by [VB], and show a relation to the results of Section 3. In this section, $\mathcal{A}$ is an abelian category, $\mathcal{D} = \mathsf{D}^*(\mathcal{A})$, where $* =$ nothing, $+, -$ or b, and

$$\mathcal{D}^{\leq 0} := \{X \in \mathcal{D} \mid \mathrm{H}^i(X) = O \text{ for } i > 0\}$$
$$\mathcal{D}^{\geq 0} := \{X \in \mathcal{D} \mid \mathrm{H}^i(X) = O \text{ for } i < 0\}$$

Definition 4.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory for $\mathcal{A}$. We set*

$${}^p\mathcal{D}^{\leq 0} := \{X \in \mathcal{D}^{\leq 0} \mid \mathrm{H}^0(X) \in \mathcal{X}\}$$
$${}^p\mathcal{D}^{\geq 0} := \{X \in \mathcal{D}^{\geq -1} \mid \mathrm{H}^{-1}(X) \in \mathcal{Y}\}$$

Lemma 4.2. *For $X^\bullet \in \mathcal{D}^{\leq 0}$, we have a distinguished triangle*

$$X^\bullet_1 \to X^\bullet \to X^\bullet_2 \to X^\bullet_1[1]$$

with $X^\bullet_1 \in {}^p\mathcal{D}^{\leq 0}$, $X^\bullet_2 \in {}^p\mathcal{D}^{\geq 1} \cap \mathcal{D}^0$.

Sketch. We have

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^{-1} & \longrightarrow & X'^{0} & \dashrightarrow\!\!\!\!\rightarrow & \tau\, \mathrm{H}^0 X^{\cdot} \\
 & & \| & & \downarrow & PB & \downarrow \\
\cdots & \longrightarrow & X^{-1} & \longrightarrow & X^{0} & \dashrightarrow\!\!\!\!\rightarrow & \mathrm{H}^0 X^{\cdot} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & O & \longrightarrow & \pi\, \mathrm{H}^0 X^{\cdot} & = = = & \pi\, \mathrm{H}^0 X^{\cdot}
\end{array}$$

□

Proposition 4.3. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory for $\mathcal{A}$. Then $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ is a non-degenerate t-structure in $\mathcal{D}$.*

Proof. For $X^{\cdot} \in {}^p\mathcal{D}^{\leq 0}$, $Y^{\cdot} \in {}^p\mathcal{D}^{\geq 1}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(X^{\cdot}, Y^{\cdot}) &\cong \mathrm{Hom}_{\mathcal{D}}(\sigma_{\geq 0} X^{\cdot}, \sigma_{\leq 0} Y^{\cdot}) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{H}^0 X^{\cdot}, \mathrm{H}^0 Y^{\cdot}) \\
&= 0
\end{aligned}$$

It is easy to see that ${}^p\mathcal{D}^{\leq 0} \subset {}^p\mathcal{D}^{\leq 1}$ and ${}^p\mathcal{D}^{\geq 1} \subset {}^p\mathcal{D}^{\geq 0}$. Let $Y^{\cdot} \in \mathcal{D}$. By Lemma 4.2, we have a commutative diagram

$$\begin{array}{ccccccc}
Y_1^{\cdot} & \longrightarrow & \tau_{\leq 0} Y^{\cdot} & \longrightarrow & Y_2^{\cdot} & \longrightarrow & Y_1^{\cdot}[1] \\
\| & & \downarrow & & \downarrow & & \| \\
Y_1^{\cdot} & \longrightarrow & Y^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & Y_1^{\cdot}[1] \\
 & & \downarrow & & \downarrow & & \\
 & & \tau_{\geq 1} Y^{\cdot} & = & \tau_{\geq 1} Y^{\cdot} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \tau_{\leq 0} Y^{\cdot}[1] & \longrightarrow & Y_2^{\cdot}[1] & &
\end{array}$$

where all vertical and horizontal sequences are distinguished triangles, and $Y_1^{\cdot} \in {}^p\mathcal{D}^{\leq 0}$, $Y_2^{\cdot} \in {}^p\mathcal{D}^{\geq 1} \cap \mathcal{D}^0$. Therefore $Z^{\cdot} \in \mathcal{D}^{\geq 0}$ and $\mathrm{H}^0 Z^{\cdot} \cong \mathrm{H}^0 Y_2^{\cdot} \in \mathcal{Y}$. Hence $Z^{\cdot} \in {}^p\mathcal{D}^{\geq 1}$. Since ${}^p\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ and ${}^p\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq -1}$, it is non-degenerate. □

Proposition 4.4. *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion theory for $\mathcal{A}$, ${}^p\mathcal{C} = {}^p\mathcal{D}^{\leq 0} \cap {}^p\mathcal{D}^{\geq 0}$. Then ${}^p\mathcal{C}$ is admissible abelian and $(\mathcal{Y}[1], \mathcal{X})$ is a torsion theory for ${}^p\mathcal{C}$.*

Proof. It is easy to see that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{Y}[1], \mathcal{X}) = \{O\}$. $X^{\cdot} \in {}^p\mathcal{C}$ iff $X^{\cdot} \cong Y^{\cdot} : Y^{-1} \to Y^0$ with $\mathrm{H}^0 Y^{\cdot} \in \mathcal{X}$ and $\mathrm{H}^{-1} Y^{\cdot} \in \mathcal{Y}$. Then we have a distinguished triangle

$$\mathrm{H}^{-1} Y^{\cdot}[1] \to Y^{\cdot} \to \mathrm{H}^0 Y^{\cdot} \to \mathrm{H}^{-1} Y^{\cdot}[2].$$

This means that we have an exact sequence in ${}^p\mathcal{C}$

$$O \to F \to Y^{\cdot} \to T \to O$$

with $F \in \mathcal{Y}[1]$, $T \in \mathcal{X}$. □

Proposition 4.5. *Let $P^{\cdot}$ be a complex $P^{-1} \to P^0$ with the P^i being small projective objects of $\mathcal{A}$. Assume $\mathcal{X}(P^{\cdot}) \cap \mathcal{Y}(P^{\cdot}) = \{0\}$ and $\mathrm{H}^0(P^{\cdot}) \in \mathcal{X}(P^{\cdot})$. Then a perverse t-structure $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ coincides with $(\mathcal{D}^{\leq 0}(P^{\cdot}), \mathcal{D}^{\geq 0}(P^{\cdot}))$.*

Proof. By Lemma 3.2. □

5. DG-Algebras and Derived Equivalences

Definition 5.1. *A differential graded algebra (a DG algebra) B over a commutative ring k is a $\mathbb{Z}$-graded k-algebra $B = \coprod_{n\in\mathbb{Z}} B^n$ endowed with a differential $d : B^n \to B^{n+1}$ $(n \in \mathbb{Z})$ such that*

$$d(ab) = d(a)b + (-1)^p ad(b)$$

for $a \in B^p$.

A DG (right) B-module M is a $\mathbb{Z}$-graded B-module $M = \coprod_{n\in\mathbb{Z}} M^n$ endowed with a differential $d : M^n \to M^{n+1}$ $(n \in \mathbb{Z})$ such that

$$d(ma) = d(m)a + (-1)^p md(a)$$

for $m \in M^p$, $a \in B$.

For DG B-module M, N and $n \in \mathbb{Z}$,

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{Gr}\,B}(M,N)^n &= \text{ the set of graded } B\text{-homomorphisms of degree } n \\
\operatorname{Hom}_{\mathsf{Gr}\,B}(M,N) &= \coprod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{Gr}\,B}(M,N)^n \\
\operatorname{Hom}_{\mathsf{Dif}\,B}(M,N) &= \operatorname{Hom}_{\mathsf{Gr}\,B}(M,N) \text{ endowed with the differential} \\
&\quad \partial : \operatorname{Hom}_{\mathsf{Gr}\,B}(M,N)^n \to \operatorname{Hom}_{\mathsf{Gr}\,B}(M,N)^{n+1} \\
(\partial((f^p)_{p\in\mathbb{Z}}) &= (d_N^{p+n} \circ f^p + (-1)^{n+1} f^{p+1} \circ d_M^p)_{p\in\mathbb{Z}}) \\
\operatorname{Hom}_{\mathcal{C}B}(M,N) &= \mathrm{Z}^0 \operatorname{Hom}_{\mathsf{Dif}\,B}(M,N) \\
\operatorname{Hom}_{\mathcal{H}B}(M,N) &= \mathrm{H}^0 \operatorname{Hom}_{\mathsf{Dif}\,B}(M,N)
\end{aligned}$$

Definition 5.2. *The suspension functor $S : \mathcal{C}B \to \mathcal{C}B$ is defined by*

$$\begin{aligned}
(SM)^n &= M^{n+1} \\
m \cdot a &= ma \\
d_{SM}^n &= -d_M^{n+1}
\end{aligned}$$

for $M \in \mathcal{C}B$.

For $u : M \to N$ in $\mathcal{C}B$, the mapping cone $\mathrm{M}(u)$ is defined by

$$\begin{aligned}
\mathrm{M}^n(u) &= N^n \oplus M^{n+1} \\
\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \cdot a &= \left[\begin{smallmatrix} na \\ ma \end{smallmatrix}\right] \\
d_{\mathrm{M}(u)}^n &= \begin{bmatrix} d_N^n & u^{n+1} \\ 0 & -d_M^{n+1} \end{bmatrix}
\end{aligned}$$

Proposition 5.3. *The following hold.*

1. *Let $\mathcal{S}_B$ be the collection of exact sequences $O \to X \to Y \to Z \to O$ in $\mathcal{C}B$ such that $O \to X^n \to Y^n \to Z^n \to O$ is split exact in* $\mathsf{Mod}\,k$. *Then $(\mathcal{C}B, \mathcal{S}_B)$ is a Frobenius category.*
2. *Let $\mathcal{T}_B$ be the collection of sextuples (X, Y, Z, i, v, w) which are isomorphic to standard triangles in $\mathcal{H}B$. Then $(\mathcal{H}B, \mathcal{T}_B)$ is a triangulated category.*

Concerning the notion of Frobenius categories, see [Ha], [Mi] Section 5.

Definition 5.4. *For a DG algebra B, $\mathrm{H}^{\cdot} B = \coprod_{n\in\mathbb{Z}} \mathrm{H}^n B$. For DG B-module M, $\mathrm{H}^{\cdot} M = \coprod_{n\in\mathbb{Z}} \mathrm{H}^n M$. Then we have the functor $\mathrm{H}^{\cdot} : \mathcal{H}A \to \mathsf{Gr}\,\mathrm{H}^{\cdot} B$. A morphism $f : M \to N$ is called quasi-isomorphism if $\mathrm{H}^{\cdot} f$ is isomorphism.*

Let Σ be the collection of quasi-isomorphisms in $\mathcal{H}B$, then $\mathcal{D}B$ is $\Sigma^{-1}\mathcal{H}B$. In this case, the canonical functor $\mathcal{C}B \to \mathcal{H}B \to \mathcal{D}B$ commutes with coproducts.

Lemma 5.5. *Let $(\mathcal{F}_i, \mathcal{S}_i)$ be Frobenius categories ($i = 1, 2$). If a functor $F : \mathcal{F}_1 \to \mathcal{F}_2$ satisfies that $F(\mathcal{S}_1) \subset \mathcal{S}_2$ and that FQ is $\mathcal{S}_2$-projective for every $\mathcal{S}_1$-projective object Q, then F induces ∂-functor $\underline{F} : \underline{\mathcal{F}}_1 \to \underline{\mathcal{F}}_2$.*

Definition 5.6. *Let $\mathcal{A}$ be an abelian category. For a complexes $X^\cdot, Y^\cdot \in \mathsf{C}(\mathcal{A})$, we define the complex $\operatorname{Hom}^\cdot_{\mathcal{A}}(X^\cdot, Y^\cdot)$ by*

$$\operatorname{Hom}^p_{\mathcal{A}}(X^\cdot, Y^\cdot) = \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^n, Y^{n+p})$$
$$\operatorname{Hom}^\cdot_{\mathcal{A}}(X^\cdot, Y^\cdot) = \coprod_{p\in\mathbb{Z}} \operatorname{Hom}^p_{\mathcal{A}}(X^\cdot, Y^\cdot)$$
$$d^p_{\operatorname{Hom}^\cdot_{\mathcal{A}}(X^\cdot,Y^\cdot)}((f^n)_{n\in\mathbb{Z}}) = (d^{n+p}_Y \circ f^n - (-1)^p f^{n+1} \circ d^n_X)_{n\in\mathbb{Z}}.$$

Proposition 5.7. *Let $\mathcal{A}$ be an AB4-category, $\mathcal{A}'$ thick abelian subcategory which is closed under coproducts. Let $C^\cdot \in \mathsf{C}_{\mathcal{A}'}(\mathcal{A})$, $B = \operatorname{End}^\cdot_{\mathsf{C}(\mathcal{A})}(C^\cdot)$. Then the following hold.*

1. *We have the functor $\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{C}_{\mathcal{A}'}(\mathcal{A}) \to \mathcal{C}B$.*
2. *$\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -)$ induces the ∂-functor $\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{K}_{\mathcal{A}'}(\mathcal{A}) \to \mathcal{H}B$.*
3. *If there is a triangulated full subcategory $\mathcal{L}$ of $\mathsf{K}_{\mathcal{A}'}(\mathcal{A})$ such that*
   (a) every $X^\cdot \in \mathsf{K}_{\mathcal{A}'}(\mathcal{A})$ has a quasi-isomorphic to some complex in $\mathcal{L}$,
   (b) $\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(\mathsf{K}^{\phi}_{\mathcal{A}'}(\mathcal{A}), \mathcal{L}) = 0$,
   then the ∂-functor $\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{K}_{\mathcal{A}'}(\mathcal{A}) \to \mathcal{H}B$ induces the right derived functor $\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{D}_{\mathcal{A}'}(\mathcal{A}) \to \mathcal{D}B$.

   Here $\mathsf{K}^{\phi}_{\mathcal{A}'}(\mathcal{A})$ is the full subcategory of $\mathsf{K}_{\mathcal{A}'}(\mathcal{A})$ consisting of acyclic complexes. In this case, we say that $\mathsf{K}_{\mathcal{A}'}(\mathcal{A})$ has a $\mathsf{K}^{\phi}_{\mathcal{A}'}(\mathcal{A})$-Bousfield localization.

Lemma 5.8. *Let $\mathcal{A}$ be an AB4-category. Let $C^\cdot \in \mathsf{C}(\mathcal{A})$ which is a bounded complex of small projective objects, and $B = \operatorname{End}^\cdot_{\mathsf{C}(\mathcal{A})}(C^\cdot)$. Then the following hold.*

1. *$\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -)$ commutes with coproducts.*
2. *$\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(C^\cdot, X^\cdot) \cong \operatorname{Hom}_{\mathcal{D}B}(\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, C^\cdot), \boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, X^\cdot))$.*

Lemma 5.9. *Let $\mathcal{A}$ be an AB4-category, $\mathcal{A}'$ thick abelian subcategory which is closed under coproducts. Assume that $\mathsf{K}_{\mathcal{A}'}(\mathcal{A})$ has a $\mathsf{K}^{\phi}_{\mathcal{A}'}(\mathcal{A})$-Bousfield localization. Let $C^\cdot \in \mathsf{C}_{\mathcal{A}'}(\mathcal{A})$ which is $\mathsf{K}^{\phi}_{\mathcal{A}'}(\mathcal{A})$-local and is compact in $\mathsf{D}_{\mathcal{A}'}(\mathcal{A})$, and $B = \operatorname{End}^\cdot_{\mathsf{C}(\mathcal{A})}(C^\cdot)$. Then the following hold.*

1. *$\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -)$ commutes with coproducts.*
2. *$\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(C^\cdot, X^\cdot) \cong \operatorname{Hom}_{\mathcal{D}B}(\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, C^\cdot), \boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, X^\cdot))$.*

Proposition 5.10. *Under the condition of Lemma 5.8 (resp., Lemma 5.9), if $\{C^\cdot[i] | i \in \mathbb{Z}\}$ is a generating set for $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}_{\mathcal{A}'}(\mathcal{A})$), then $\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{D}(\mathcal{A}) \to \mathcal{D}B$ (resp., $\boldsymbol{R}\operatorname{Hom}^\cdot_{\mathcal{A}}(C^\cdot, -) : \mathsf{D}_{\mathcal{A}'}(\mathcal{A}) \to \mathcal{D}B$) is an equivalence.*

Proof. By Theorem 6.3 of Appendix. □

Corollary 5.11. *Let $P^\cdot$ be a bounded complex of finitely generated projective modules over a ring A, $B = \operatorname{End}^\cdot_{\mathsf{C}(\mathcal{A})}(C^\cdot)$. If $\{P^\cdot[i] | i \in \mathbb{Z}\}$ is a generating set for $\mathsf{D}(\mathsf{Mod}\, A)$, then $\boldsymbol{R}\operatorname{Hom}^\cdot_{A}(P^\cdot, -) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathcal{D}B$ is an equivalence.*

Corollary 5.12. *Let X be a quasi-compact separated scheme over an algebraically closed field. If a perfect complex $C^\cdot \in \mathsf{C}^+_{qc}(\mathsf{Inj}\, X)$ satisfies that $\{C^\cdot[i] | i \in \mathbb{Z}\}$ is a generating set for $\mathsf{D}_{qc}(X)$ (or $\mathsf{D}(\mathsf{QCoh}\, X)$), then*

$$\mathsf{D}(\mathsf{QCoh}\, X) \cong \mathcal{D}B$$

with $B = \mathrm{End}^{\bullet}_{\mathsf{C}(\mathcal{A})}(C^{\bullet})$.

Proof. According to [BN], $\mathsf{K}_{qc}(X)$ has a $\mathsf{K}^{\phi}_{qc}(X)$-Bousfield localization, and $\mathsf{D}_{qc}(X) \cong \mathsf{D}(\mathsf{QCoh}\,X)$. By 5.10 we complete the proof. □

Corollary 5.13. *Let X be a projective scheme which embeds to $\mathbf{P}^n_k$. If a complex $C^{\bullet} \in \mathsf{C}^{+}_{qc}(\mathsf{Inj}\,X)$ which is quasi-isomorphic to $\oplus_{i=0}^{n}\mathcal{O}_X(-i)$, then*

$$\mathsf{D}(\mathsf{QCoh}\,X) \cong \mathcal{D}B.$$

Sketch. Let V be an $(n+1)$-dimensional k-vector space. In $\mathsf{Mod}\,\mathbf{P}^n_k$ we have an exact sequence

$$\begin{aligned} O \to \wedge^{n+1}V \otimes \mathcal{O}_{\mathbf{P}}(-n-1) \to \wedge^{n}V \otimes \mathcal{O}_{\mathbf{P}}(-n) \to \dots \\ \to \wedge^{1}V \otimes \mathcal{O}_{\mathbf{P}}(-1) \to \mathcal{O}_{\mathbf{P}} \to O \end{aligned}$$

Since the above sequence is locally split exact, we have an exact sequence in $\mathsf{Mod}\,X$

$$\begin{aligned} O \to \wedge^{n+1}V \otimes \mathcal{O}_{X}(-n-1) \to \wedge^{n}V \otimes \mathcal{O}_{X}(-n) \to \dots \\ \to \wedge^{1}V \otimes \mathcal{O}_{X}(-1) \to \mathcal{O}_{X} \to O \end{aligned}$$

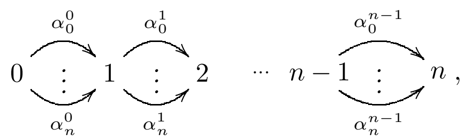
Therefore $\oplus_{i=0}^{n}\mathcal{O}_X(-i)$ generates $\mathsf{D}(\mathsf{Qcoh}\,X)$. □

Corollary 5.14 ([Be])**.** *Let $B' = \mathrm{End}_{\mathbf{P}^n_k}(\oplus_{i=0}^{n}\mathcal{O}_{\mathbf{P}^n}(-i))$, then*

$$\mathsf{D}(\mathsf{QCoh}\,\mathbf{P}^n_k) \cong \mathsf{D}(\mathsf{Mod}\,B')$$
$$\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{P}^n_k) \cong \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B').$$

Proof. By Corollary 5.13, we have $\mathsf{D}(\mathsf{QCoh}\,\mathbf{P}^n_k) \cong \mathcal{D}B$. Let $B'' = \sigma_{\leq 0}B$ and $B' = \mathrm{H}^0 B$, we have morphisms $B' \leftarrow B'' \rightarrow B$ which induce $B' \overset{\sim}{\leftarrow} \mathrm{H}^{\bullet} B'' \overset{\sim}{\rightarrow} \mathrm{H}^{\bullet} B$. By [Ke1] 6.1 Example, we have $\mathsf{D}(\mathsf{QCoh}\,\mathbf{P}^n_k) \cong \mathsf{D}(\mathsf{Mod}\,B')$. Since $\mathsf{Coh}\,\mathbf{P}^n_k$ and $\mathsf{mod}\,B'$ have finite global dimensions, the full subcategories of $\mathsf{D}(\mathsf{QCoh}\,\mathbf{P}^n_k)$ and $\mathsf{D}(\mathsf{Mod}\,B')$ consisting of compact objects are equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{P}^n_k)$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,B')$, respectively. By Theorem 6.3, we complete the proof. □

Remark 5.15. *Let $(\vec{Q}, \rho)$ be the following quiver with relations:*



and ρ is the set of relations over k

$$\alpha_i^{l+1}\alpha_j^{l} = \alpha_j^{l+1}\alpha_i^{l} \text{ for } 0 \leq i < j \leq n, 0 \leq l < n-1.$$

Then B' of Corollary 5.14 is isomorphic to $k(\vec{Q}, \rho)$.

Remark 5.16. *Recently, Bondal and Van den Bergh showed that the derived category $\mathsf{D}(\mathsf{QCoh}\,X)$ of quasi-coherent sheaves of a Noetherian scheme X has a compact generator. By using [Ke2], they also showed that $\mathsf{D}(\mathsf{QCoh}\,X) \cong \mathcal{D}B$ for some DG algebra B.*

Example 5.17. *In Example 3.10, let $B' = \mathrm{End}^{\bullet}_{A}(P^{\bullet})$. Then we have*

$$\mathsf{D}(\mathsf{Mod}\,A) \cong \mathcal{D}B'.$$

Let $B'' = B^{-1} \oplus B^0$ *with*

$$B^{-1} \to B^0 : \mathrm{Hom}_A(P^0, P^{-1}) \to \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\,A)}(P^\bullet, P^\bullet)$$
$$(f \mapsto (f \circ d^{-1} - f^{-1} \circ f))$$

According to [Ke1] *6.1 Example, the natural inclusion* $B'' \to B'$ *induces the derived equivalence* $\mathcal{D}B'' \cong \mathcal{D}B'$*. Hence we have*

$$\mathsf{D}(\mathsf{Mod}\,A) \cong \mathcal{D}B''.$$

Example 5.18. *In Proposition 5.13, let* $B' = \mathrm{End}^\bullet_A(P^\bullet)$*. Then we have*

$$\mathsf{D}(\mathsf{Mod}\,A) \cong \mathcal{D}B'.$$

Let $B'' = B^{-1} \oplus B^0$ *with*

$$B^{-1} \to B^0 : \mathrm{Hom}_A(P^0, P^{-1}) \to \mathrm{Hom}_{\mathsf{C}(\mathsf{Mod}\,A)}(P^\bullet, P^\bullet)$$
$$(f \mapsto (f \circ d^{-1} - f^{-1} \circ f))$$

According to [Ke1] *6.1 Example, the natural inclusion* $B'' \to B'$ *induces the derived equivalence* $\mathcal{D}B'' \cong \mathcal{D}B'$*. Hence we have*

$$\mathsf{D}(\mathsf{Mod}\,A) \cong \mathcal{D}B''.$$

6. Appendix

Throughout this section all triangulated categories contains arbitrary coproducts.

Definition 6.1. *A triangulated full subcategory* $\mathcal{L}$ *of* $\mathcal{T}$ *is called localizing provided that every coproduct of objects in* $\mathcal{L}$ *is in* $\mathcal{L}$*.*

Lemma 6.2. *Let* $\mathcal{T}$ *be a triangulated category,* $\mathcal{S}$ *a generating set. Let* $\mathcal{L}$ *be a localizing subcategory of* $\mathcal{T}$ *which contains* $\mathcal{S}$*. Then* $\mathcal{L} = \mathcal{T}$*. Furthermore, for every* $X \in \mathcal{T}$*, there are distinguished triangles*

$$Z_n \to X_n \to X_{n+1} \to Z_n[1]$$

with $X_0, Z_n \in \mathsf{Sum}\,\mathcal{S}$ $(n \geq 0)$*, such that*

$$X \cong \underrightarrow{\mathrm{hlim}}\, X_n$$

Here $\mathsf{Sum}\,\mathcal{S}$ *is the full subcategory of* $\mathcal{T}$ *consisting of coproducts of objects* $X \in \mathcal{S}$*.*

Proof. See [Ke1] 5.2 Theorem and [Ne] Theorem 4.1. □

Theorem 6.3. *Let* $F : \mathcal{T}_1 \to \mathcal{T}_2$ *be a* ∂*-functor commuting with coproducts. Assume that there is a generating set* $\mathcal{S}$ *for* $\mathcal{T}_1$ *such that* $F\mathcal{S}$ *is a generating set for* $\mathcal{T}_2$*. If* $F|_{\mathcal{S}}$ *is fully faithful, then* $F : \mathcal{T}_1 \to \mathcal{T}_2$ *is an triangle equivalence. In this case,* F *induces the triangle equivalence* $\mathcal{T}_1^c \to \mathcal{T}_2^c$*, where* $\mathcal{T}_i^c$ *is the triangulated full subcategory of* $\mathcal{T}_i$ *consisting of compact objects.*

Proof. Step 1. We have $\mathrm{Hom}_{\mathcal{T}_1}(C, Y) \cong \mathrm{Hom}_{\mathcal{T}_2}(FC, FY)$ for $C \in \mathcal{S}$ and $Y \in \mathsf{Sum}\,\mathcal{S}$.

Step 2. We have $\mathrm{Hom}_{\mathcal{T}_1}(C, Y) \cong \mathrm{Hom}_{\mathcal{T}_2}(FC, FY)$ for $C \in \mathcal{S}$ and $Y \in \mathcal{T}_1$.

$\because$) Given $Y \in \mathcal{T}_1$, by Lemma 6.2, there are distinguished triangles

$$Z_n \to Y_n \to Y_{n+1} \to Z_n[1]$$

with $Y_0, Z_n \in \mathsf{Sum}\,\mathcal{S}$ $(n \geq 0)$, such that

$$Y \cong \underrightarrow{\mathrm{hlim}}\, Y_n$$

By induction on n, we have $\mathrm{Hom}_{\mathcal{T}_1}(C, X_n) \cong \mathrm{Hom}_{\mathcal{T}_2}(FC, FY_n)$. Since FC is compact, we have $\mathrm{Hom}_{\mathcal{T}_1}(C, \underrightarrow{\mathrm{hlim}}\, Y_n) \cong \mathrm{Hom}_{\mathcal{T}_2}(FC, F\underrightarrow{\mathrm{hlim}}\, Y_n)$.

Step 3. We have $\mathrm{Hom}_{\mathcal{T}_1}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}_2}(FX, FY)$ for $X, Y \in \mathcal{T}_1$.

$\because$) It is similar to Step 2.

Step 4. Given $M \in \mathcal{T}_2$, by Lemma 6.2, there are distinguished triangles

$$N_n \to M_n \to M_{n+1} \to N_n[1]$$

with $M_0, N_n \in \mathsf{Sum}\, F\mathcal{S}$ $(n \geq 0)$, such that

$$M \cong \underrightarrow{\mathrm{hlim}}\, M_n$$

Since F is fully faithful, by induction there are distinguished triangles

$$Z_n \to X_n \to X_{n+1} \to Z_n[1]$$

with $X_0, Z_n \in \mathsf{Sum}\, \mathcal{S}$ $(n \geq 0)$, such that

$$\begin{array}{ccccccc} FZ_n & \longrightarrow & FX_n & \longrightarrow & FX_{n+1} & \longrightarrow & FZ_n[1] \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ N_n & \longrightarrow & M_n & \longrightarrow & M_{n+1} & \longrightarrow & N_n[1] \end{array}$$

Hence

$$\begin{aligned} M &\cong \underrightarrow{\mathrm{hlim}}\, M_n \\ &\cong F\underrightarrow{\mathrm{hlim}}\, X_n \\ &\cong FX \end{aligned}$$

Since the compactness of an object is the categorical property, the last assertion is trivial. □

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