RECOLLEMENT AND TILTING COMPLEXES

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ABSTRACT. First, we study recollement of a derived category of unbounded complexes of modules induced by a partial tilting complex. Second, we give equivalent conditions for $P^{\boldsymbol{\cdot}}$ to be a recollement tilting complex, that is, a tilting complex which induces an equivalence between recollements $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ and $\{\mathsf{D}_{B/BfB}(B), \mathsf{D}(B), \mathsf{D}(fBf)\}$, where e, f are idempotents of A, B, respectively. In this case, there is an unbounded bimodule complex $\Delta^{\boldsymbol{\cdot}}_T$ which induces an equivalence between $\mathsf{D}_{A/AeA}(A)$ and $\mathsf{D}_{B/BfB}(B)$. Third, we apply the above to a symmetric algebra A. We show that a partial tilting complex $P^{\boldsymbol{\cdot}}$ for A of length 2 extends to a tilting complex, and that $P^{\boldsymbol{\cdot}}$ is a tilting complex if and only if the number of indecomposable types of $P^{\boldsymbol{\cdot}}$ is one of A. Finally, we show that for an idempotent e of A, a tilting complex for eAe extends to a recollement tilting complex for A, and that its standard equivalence induces an equivalence between $\mathsf{Mod}\, A/AeA$ and $\mathsf{Mod}\, B/BfB$.

0. Introduction

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne in connection with derived categories of sheaves of topological spaces ([1]). In representation theory, Cline, Parshall and Scott applied this notion to finite dimensional algebras over a field, and introduced the notion of quasi-hereditary algebras ([5], [14]). In quasi-hereditary algebras, idempotents of algebras play an important role. In [16], Rickard introduced the notion of tilting complexes as a generalization of tilting modules. Many constructions of tilting complexes have a relation to idempotents of algebras (e.g. [13], [19], [6], [7]). We studied constructions of tilting complexes of term length 2 which has an application to symmetric algebras ([8]). In the case of algebras of infinite global dimension, we cannot treat recollement of derived categories of bounded complexes such as one in the case of quasi-hereditary algebras. In this paper, we study recollement of derived categories of unbounded complexes of modules for k-projective algebras over a commutative ring k, and give the conditions that tilting complexes induce equivalences between recollements induced by idempotents. Moreover, we give some constructions of tilting complexes over symmetric algebras.

In Section 2, for a k-projective algebra A over a commutative ring k, we study a recollement $\{\mathcal{K}_P, \mathsf{D}(A), \mathsf{D}(B)\}$ of a derived category $\mathsf{D}(A)$ of unbounded complexes of right A-modules induced by a partial tilting complex $P^{\boldsymbol{\cdot}}$, where $B = \mathrm{End}_{\mathsf{D}(A)}(P^{\boldsymbol{\cdot}})$. We show that there exists the triangle ξ_V in $\mathsf{D}(A^{\mathrm{e}})$ which induce adjoint functors of this recollement, and that the triangle ξ_V is isomorphic to a triangle which is constructed by a $P^{\boldsymbol{\cdot}}$-resolution of A in the sense of Rickard (Theorem 2.8, Proposition 2.15, Corollary 2.16). In general, this recollement is out of localizations of

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triangulated categories which Neeman treated in [12] (Corollary 2.9). Moreover, we study a recollement $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ which is induced by an idempotent e of A (Proposition 2.17, Corollary 2.19). In Section 3, we study equivalences between recollements which are induced by idempotents. We give equivalent conditions for $P^{\bullet}$ to be a tilting complex inducing an equivalence between recollements $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ and $\{\mathsf{D}_{B/BfB}(B), \mathsf{D}(B), \mathsf{D}(fBf)\}$ (Theorem 3.5). We call this tilting complex a recollement tilting complex related to an idempotent e. There are many symmetric properties between algebras A and B for a two-sided recollement tilting complex ${}_BT^{\bullet}_A$(Corollaries 3.7, 3.8). Moreover, we have an unbounded bimodule complex $\mathit{\Delta}^{\bullet}_T \in \mathsf{D}(B^{\circ} \otimes A)$ which induces an equivalence between $\mathsf{D}_{A/AeA}(A)$ and $\mathsf{D}_{B/BfB}(B)$. The complex $\mathit{\Delta}^{\bullet}_T$ is a compact object in $\mathsf{D}_{A/AeA}(A)$, and satisfies properties such as a tilting complex (Propositions 3.11, 3.13, 3.14, Corollary 3.12). In Section 4, we study constructions of tilting complexes for a symmetric algebra A over a field. First, we construct a family of complexes $\{\Theta^{\bullet}_n(P^{\bullet}, A)\}_{n\geq 0}$ from a partial tilting complex $P^{\bullet}$, and give equivalent conditions for $\Theta^{\bullet}_n(P^{\bullet}, A)$ to be a tilting complex (Definition 4.3, Theorem 4.6, Corollary 4.7). As applications, we show that a partial tilting complex $P^{\bullet}$ of length 2 extends to a tilting complex, and that $P^{\bullet}$ is a tilting complex if and only if the number of indecomposable types of $P^{\bullet}$ is one of A (Corollaries 4.8, 4.9). This is a complex version over symmetric algebras of Bongartz's result on classical tilting modules ([3]). Second, for an idempotent e of A, by the above construction a tilting complex for eAe extends to a recollement tilting complex $T^{\bullet}$ related to e (Theorem 4.11). This recollement tilting complex induces that A/AeA is isomorphic to B/BfB as a ring, and that the standard equivalence $\boldsymbol{R}\mathrm{Hom}^{\bullet}_A(T^{\bullet}, -)$ induces an equivalence between $\mathsf{Mod}\, A/AeA$ and $\mathsf{Mod}\, B/BfB$ (Corollary 4.12). This construction of tilting complexes contains constructions obtained by several authors.

1. Basic Tools on k-Projective Algebras

In this section, we recall basic tools of derived functors in the case of k-projective algebras over a commutative ring k. Throughout this section, we deal only with k-projective k-algebras, that is, k-algebras which are projective as k-modules. For a k-algebra A, we denote by $\mathsf{Mod}\, A$ the category of right A-modules, and denote by $\mathsf{Proj}\, A$ (resp., $\mathsf{proj}\, A$) the full additive subcategory of $\mathsf{Mod}\, A$ consisting of projective (resp., finitely generated projective) modules. For an abelian category $\mathcal{A}$ and an additive category $\mathcal{B}$, we denote by $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}^{+}(\mathcal{A})$, $\mathsf{D}^{-}(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes of $\mathcal{A}$ (resp., complexes of $\mathcal{A}$ with bounded below cohomologies, complexes of $\mathcal{A}$ with bounded above cohomologies, complexes of $\mathcal{A}$ with bounded cohomologies), denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded complexes) of $\mathcal{B}$ (see [15] for details). In the case of $\mathcal{A} = \mathcal{B} = \mathsf{Mod}\, A$, we simply write $\mathsf{K}^{*}(A)$ and $\mathsf{D}^{*}(A)$ for $\mathsf{K}^{*}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{*}(\mathsf{Mod}\, A)$, respectively. Given a k-algebra A we denote by A° the opposite algebra, and by A^{e} the enveloping algebra $A^{\circ} \otimes_k A$. We denote by $Res_A : \mathsf{Mod}\, B^{\circ} \otimes_k A \to \mathsf{Mod}\, A$ the forgetful functor, and use the same symbol $Res_A : \mathsf{D}(B^{\circ} \otimes_k A) \to \mathsf{D}(A)$ for the induced derived functor. Throughout this paper, we simply write $\otimes$ for $\otimes_k$.

In the case of k-projective k-algebras A, B and C, using [4] Chapter IX Section 2, we don't need to distinguish the derived functor

$$Res_k \circ (\boldsymbol{R}\operatorname{Hom}^{\bullet}_C) : \mathsf{D}(A^\circ \otimes C)^\circ \times \mathsf{D}(B^\circ \otimes C) \to \mathsf{D}(B^\circ \otimes A) \to \mathsf{D}(k)$$

$$(\text{resp., } Res_k \circ (\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B) : \mathsf{D}(A^\circ \otimes B) \times \mathsf{D}(B^\circ \otimes C) \to \mathsf{D}(A^\circ \otimes C) \to \mathsf{D}(k))$$

with the derived functor

$$\boldsymbol{R}\operatorname{Hom}^{\bullet}_C \circ ((Res_C)^\circ \times Res_C) : \mathsf{D}(A^\circ \otimes C)^\circ \times \mathsf{D}(B^\circ \otimes C) \to \mathsf{D}(C)^\circ \times \mathsf{D}(C) \to \mathsf{D}(k)$$

$$(\text{resp., } \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \circ (Res_B \times Res_{B^\circ}) : \mathsf{D}(A^\circ \otimes B) \times \mathsf{D}(B^\circ \otimes C) \to \mathsf{D}(B) \times \mathsf{D}(B^\circ) \to \mathsf{D}(k))$$

(see [17], [2] and [20] for details). We freely use this fact in this paper. Moreover, we have the following statements.

Proposition 1.1. *Let k be a commutative ring, A, B, C, D k-projective k-algebras. The following hold.*

(1) *For ${}_AU^{\bullet}_B \in \mathsf{D}(A^\circ \otimes B)$, ${}_BV^{\bullet}_C \in \mathsf{D}(B^\circ \otimes C)$, ${}_CW^{\bullet}_D \in \mathsf{D}(C^\circ \otimes D)$, we have an isomorphism in $\mathsf{D}(A^\circ \otimes D)$:*

$$({}_AU^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_C W^{\bullet}_D \cong {}_AU^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B (V^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_C W^{\bullet}_D).$$

(2) *For ${}_AU^{\bullet}_B \in \mathsf{D}(A^\circ \otimes B)$, ${}_DV^{\bullet}_C \in \mathsf{D}(D^\circ \otimes C)$, ${}_AW^{\bullet}_C \in \mathsf{D}(D^\circ \otimes C)$, we have an isomorphism in $\mathsf{D}(B^\circ \otimes D)$:*

$$\boldsymbol{R}\operatorname{Hom}^{\bullet}_A({}_AU^{\bullet}_B, \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_DV^{\bullet}_C, {}_AW^{\bullet}_C)) \cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_DV^{\bullet}_C, \boldsymbol{R}\operatorname{Hom}^{\bullet}_A({}_AU^{\bullet}_B, {}_AW^{\bullet}_C)).$$

(3) *For ${}_AU^{\bullet}_B \in \mathsf{D}(A^\circ \otimes B)$, ${}_BV^{\bullet}_C \in \mathsf{D}(B^\circ \otimes C)$, ${}_DW^{\bullet}_C \in \mathsf{D}(D^\circ \otimes C)$, we have an isomorphism in $\mathsf{D}(D^\circ \otimes A)$:*

$$\boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_AU^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}_C, {}_DW^{\bullet}_C) \cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B({}_AU^{\bullet}_B, \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_BV^{\bullet}_C, {}_DW^{\bullet}_C)).$$

(4) *For ${}_AU^{\bullet}_B \in \mathsf{D}(A^\circ \otimes B)$, ${}_BV^{\bullet}_C \in \mathsf{D}(B^\circ \otimes C)$, ${}_AW^{\bullet}_C \in \mathsf{D}(A^\circ \otimes C)$, we have an isomorphism in $\mathsf{D}(k)$:*

$$\boldsymbol{R}\operatorname{Hom}^{\bullet}_{A^\circ \otimes C}({}_AU^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}_C, {}_AW^{\bullet}_C) \cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_{A^\circ \otimes B}({}_AU^{\bullet}_B, \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_BV^{\bullet}_C, {}_AW^{\bullet}_C)).$$

(5) *For ${}_AU^{\bullet}_B \in \mathsf{D}(A^\circ \otimes B)$, ${}_BV^{\bullet}_C \in \mathsf{D}(B^\circ \otimes C)$, ${}_AW^{\bullet}_C \in \mathsf{D}(A^\circ \otimes C)$, we have a commutative diagram:*

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}(A^\circ \otimes C)}({}_AU^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}_C, {}_AW^{\bullet}_C) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathsf{D}(A^\circ \otimes B)}({}_AU^{\bullet}_B, \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_BV^{\bullet}_C, {}_AW^{\bullet}_C)) \\
Res_C \downarrow & & \downarrow Res_B \\
\operatorname{Hom}_{\mathsf{D}(C)}(U^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}_C, W^{\bullet}_C) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathsf{D}(B)}(U^{\bullet}_B, \boldsymbol{R}\operatorname{Hom}^{\bullet}_C({}_BV^{\bullet}_C, W^{\bullet}_C)),
\end{array}$$

where all horizontal arrows are isomorphisms induced by 3 and 4. Equivalently, we don't need to distinguish the adjunction arrows induced by ${}_BV^{\bullet}_C$ (see [10]*, IV, 7).*

Definition 1.2. *A complex $X^{\bullet} \in \mathsf{D}(A)$ is called a perfect complex if $X^{\bullet}$ is isomorphic to a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ in $\mathsf{D}(A)$. We denote by $\mathsf{D}(A)_{\mathrm{perf}}$ the triangulated full subcategory of $\mathsf{D}(A)$ consisting of perfect complexes. A bimodule complex $X^{\bullet} \in \mathsf{D}(B^\circ \otimes_k A)$ is called a biperfect complex if $Res_A(X^{\bullet}) \in \mathsf{D}(A)_{\mathrm{perf}}$ and if $Res_{B^\circ}(X^{\bullet}) \in \mathsf{D}(B^\circ)_{\mathrm{perf}}$.*

For an object C of a triangulated category $\mathcal{D}$, C is called a compact object in $\mathcal{D}$ if $\operatorname{Hom}_{\mathcal{D}}(C, -)$ commutes with arbitrary coproducts on $\mathcal{D}$.

For a complex $X^\cdot = (X^i, d^i)$, we define the following truncations:

$$\sigma_{\leq n}X^\cdot : \ldots \to X^{n-2} \to X^{n-1} \to \operatorname{Ker} d^n \to 0 \to \ldots,$$
$$\sigma'_{\geq n}X^\cdot : \ldots \to 0 \to \operatorname{Cok} d^{n-1} \to X^{n+1} \to X^{n+2} \to \ldots.$$

The following characterization of perfect complexes is well known (cf. [16]). For the convenience of the reader, we give a simple proof.

Proposition 1.3. *For $X^\cdot \in \mathsf{D}(A)$, the following are equivalent.*

(1) *$X^\cdot$ is a perfect complex.*
(2) *$X^\cdot$ is a compact object in $\mathsf{D}(A)$.*

Proof. $1 \Rightarrow 2$. It is trivial, because we have isomorphisms:

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D}(A)}(X^\cdot, -) &\cong R^0 \operatorname{Hom}^\cdot_A(X^\cdot, -)\\ &\cong \mathrm{H}^0(- \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_A \boldsymbol{R}\operatorname{Hom}^\cdot_A(X^\cdot, A)).\end{aligned}$$

$2 \Rightarrow 1$. According to [2] or [20], there is a complex $P^\cdot : \ldots \to P^{n-1} \xrightarrow{d^{n-1}} P^n \to \ldots \in \mathsf{K}(\mathsf{Proj}\, A)$ such that

(a) $P^\cdot \cong X^\cdot$ in $\mathsf{D}(A)$,
(b) $\operatorname{Hom}_{\mathsf{K}(A)}(P^\cdot, -) \cong \operatorname{Hom}_{\mathsf{D}(A)}(P^\cdot, -)$.

Consider the complex $C^\cdot : \ldots \xrightarrow{0} \operatorname{Cok} d^{n-1} \xrightarrow{0} \ldots$, then it is easy to see that $C^\cdot =$ the coproduct $\coprod_{n\in\mathbb{Z}} \operatorname{Cok} d^{n-1}[-n] =$ the product $\prod_{n\in\mathbb{Z}} \operatorname{Cok} d^{n-1}[-n]$, that is the biproduct $\bigoplus_{n\in\mathbb{Z}} \operatorname{Cok} d^{n-1}[-n]$ of $\operatorname{Cok} d^{n-1}[-n]$. Since we have isomorphisms in $\mathsf{Mod}\, k$:

$$\begin{aligned}\coprod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{K}(A)}(P^\cdot, \operatorname{Cok} d^{n-1}[-n]) &\cong \operatorname{Hom}_{\mathsf{K}(A)}(P^\cdot, \bigoplus_{n\in\mathbb{Z}} \operatorname{Cok} d^{n-1}[-i])\\ &\cong \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{K}(A)}(P^\cdot, \operatorname{Cok} d^{n-1}[-n]),\end{aligned}$$

it is easy to see $\operatorname{Hom}_{\mathsf{K}(A)}(P^\cdot, \operatorname{Cok} d^{n-1}[-n]) = 0$ for all but finitely many $n \in \mathbb{Z}$. Then there are $m \leq n$ such that $P^\cdot \cong \sigma'_{\geq m}\sigma_{\leq n}P^\cdot$ and $\sigma'_{\geq m}\sigma_{\leq n}P^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$. According to [16] Proposition 6.3, we complete the proof. □

Definition 1.4. *We call a complex $X^\cdot \in \mathsf{D}(A)$ a partial tilting complex if*

(a) *$X^\cdot \in \mathsf{D}(A)_{\mathrm{perf}}$,*
(b) *$\operatorname{Hom}_{\mathsf{D}(A)}(X^\cdot, X^\cdot[n]) = 0$ for all $n \neq 0$.*

Definition 1.5. *Let $X^\cdot \in \mathsf{D}(A)$ be a partial tilting complex, and $B = \operatorname{End}_{\mathsf{D}(A)}(X^\cdot)$. According to [9] Theorem, there exists a unique bimodule complex $V^\cdot \in \mathsf{D}(B^\circ \otimes A)$ up to isomorphism such that*

(a) *there is an isomorphism $\phi : X^\cdot \xrightarrow{\sim} Res_A V^\cdot$ in $\mathsf{D}(A)$ such that $\phi f = \lambda_B(f)\phi$ for any $f \in \operatorname{End}_{\mathsf{D}(A)}(X^\cdot)$, where $\lambda_B : B \to \operatorname{End}_{\mathsf{D}(A)}(V^\cdot)$ is the left multiplication morphism.*

We call $V^\cdot$ the associated bimodule complex of $X^\cdot$. In this case, the left multiplication morphism $\lambda_B : B \to \boldsymbol{R}\operatorname{Hom}^\cdot_A(V^\cdot, V^\cdot)$ is an isomorphism in $\mathsf{D}(B^{\mathrm{e}})$.

Rickard showed that for a tilting complex $P^\cdot$ in $\mathsf{D}(A)$ with $B = \operatorname{End}_{\mathsf{D}(A)}(P^\cdot)$, there exists a two-sided tilting complex ${}_BT^\cdot_A \in \mathsf{D}(B^\circ \otimes A)$ ([17]).

Definition 1.6. *A bimodule complex ${}_BT^\cdot_A \in \mathsf{D}(B^\circ \otimes_k A)$ is called a two-sided tilting complex provided that*

(a) ${}_BT^{\boldsymbol{\cdot}}_A$ *is a biperfect complex.*
(b) *There exists a biperfect complex* ${}_AT^{\vee\boldsymbol{\cdot}}_B$ *such that*
(b1) ${}_BT^{\boldsymbol{\cdot}} \overset{\boldsymbol{\cdot}}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\boldsymbol{\cdot}}_B \cong B$ *in* $\mathsf{D}(B^{\mathrm{e}})$,
(b2) ${}_AT^{\vee\boldsymbol{\cdot}} \overset{\boldsymbol{\cdot}}{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}}_A \cong A$ *in* $\mathsf{D}(A^{\mathrm{e}})$.

We call ${}_AT^{\vee\boldsymbol{\cdot}}_B$ *the inverse of* ${}_BT^{\boldsymbol{\cdot}}_A$.

Proposition 1.7 ([17]). *For a two-sided tilting complex* ${}_BT^{\boldsymbol{\cdot}}_A \in \mathsf{D}(B^\circ \otimes A)$, *the following hold.*

(1) *We have isomorphisms in* $\mathsf{D}(A^\circ \otimes B)$:

$$\begin{aligned} {}_AT^{\vee\boldsymbol{\cdot}}_B &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T, A) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T, B). \end{aligned}$$

(2) $\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}}, -) \cong - \overset{\boldsymbol{\cdot}}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\boldsymbol{\cdot}} : \mathsf{D}^*(A) \to \mathsf{D}^*(B)$ *is a triangle equivalence, and has* $\boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\vee\boldsymbol{\cdot}}, -) \cong - \overset{\boldsymbol{\cdot}}{\otimes}{}^{\boldsymbol{L}}_B T^{\boldsymbol{\cdot}} : \mathsf{D}^*(B) \to \mathsf{D}^*(A)$ *as a quasi-inverse, where* $* =$ *nothing*, $+, -, \mathrm{b}$.

In the case of k-projective k-algebras, by [17] we have also the following result (see also Lemma 2.6).

Proposition 1.8. *For a bimodule complex* ${}_BT^{\boldsymbol{\cdot}}_A$, *the following are equivalent.*

(1) ${}_BT^{\boldsymbol{\cdot}}_A$ *is a two-sided tilting complex.*
(2) ${}_BT^{\boldsymbol{\cdot}}_A$ *satisfies that*
(a) ${}_BT^{\boldsymbol{\cdot}}_A$ *is a biperfect complex,*
(b) the right multiplication morphism $\rho_A : A \to \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_B(T^{\boldsymbol{\cdot}}, T^{\boldsymbol{\cdot}})$ *is an isomorphism in* $\mathsf{D}(A^{\mathrm{e}})$,
(c) the left multiplication morphism $\lambda_B : B \to \boldsymbol{R}\operatorname{Hom}^{\boldsymbol{\cdot}}_A(T^{\boldsymbol{\cdot}}, T^{\boldsymbol{\cdot}})$ *is an isomorphism in* $\mathsf{D}(B^{\mathrm{e}})$.

2. Recollement and Partial Tilting Complexes

In this section, we study recollements of a derived category $\mathsf{D}(A)$ induced by a partial tilting complex $P^{\boldsymbol{\cdot}}_A$ and induced by an idempotent e of A. Throughout this section, all algebras are k-projective algebras over a commutative ring k.

Definition 2.1. *Let* $\mathcal{D}, \mathcal{D}''$ *be triangulated categories, and* $j^* : \mathcal{D} \to \mathcal{D}''$ *a* ∂*-functor. If* j^* *has a fully faithful right (resp., left) adjoint* $j_* : \mathcal{D}'' \to \mathcal{D}$ *(resp.,* $j_! : \mathcal{D}'' \to \mathcal{D}$*), then* $\{\mathcal{D}, \mathcal{D}''; j^*, j_*\}$ *(resp.,* $\{\mathcal{D}, \mathcal{D}''; j_!, j^*\}$*) is called a localization (resp., colocalization) of* $\mathcal{D}$*. Moreover, if* j^* *has a fully faithful right adjoint* $j_* : \mathcal{D}'' \to \mathcal{D}$ *and a fully faithful left adjoint* $j_! : \mathcal{D}'' \to \mathcal{D}$*, then* $\{\mathcal{D}, \mathcal{D}''; j_!, j^*, j_*\}$ *is called a bilocalization of* $\mathcal{D}$.

For full subcategories $\mathcal{U}$ *and* $\mathcal{V}$ *of* $\mathcal{D}$, $(\mathcal{U}, \mathcal{V})$ *is called a stable* t*-structure in* $\mathcal{D}$ *provided that*

(a) $\mathcal{U}$ *and* $\mathcal{V}$ *are stable for translations.*
(b) $\operatorname{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$.
(c) *For every* $X \in \mathcal{D}$*, there exists a triangle* $U \to X \to V \to U[1]$ *with* $U \in \mathcal{U}$ *and* $V \in \mathcal{V}$.

We have the following properties.

Proposition 2.2 ([1], cf. [11]). *Let $(\mathcal{U},\mathcal{V})$ be a stable t-structure in a triangulated category $\mathcal{D}$, and let $U\to X\to V\to U[1]$ and $U'\to X'\to V'\to U'[1]$ be triangles in $\mathcal{D}$ with $U,U'\in\mathcal{U}$ and $V,V'\in\mathcal{V}$. For any morphism $f:X\to X'$, there exist a unique $f_{\mathcal{U}}:U\to U'$ and a unique $f_{\mathcal{V}}:V\to V'$ which induce a morphism of triangles:*

$$\begin{array}{ccccccc} U & \longrightarrow & X & \longrightarrow & V & \longrightarrow & U[1] \\ {\scriptstyle f_{\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{\mathcal{V}}} & & \downarrow{\scriptstyle f_{\mathcal{U}}[1]} \\ U' & \longrightarrow & X' & \longrightarrow & V' & \longrightarrow & U'[1]. \end{array}$$

In particular, for any $X\in\mathcal{D}$, the above U and V are uniquely determined up to isomorphism.

Proposition 2.3 ([11]). *The following hold.*

(1) *If $\{\mathcal{D},\mathcal{D}'';j^*,j_*\}$ (resp., $\{\mathcal{D},\mathcal{D}'';j_!,j^*\}$) is a localization (resp., a colocalization) of $\mathcal{D}$, then $(\operatorname{Ker}j^*,\operatorname{Im}j_*)$ (resp., $(\operatorname{Im}j_!,\operatorname{Ker}j^*)$) is a stable t-structure. In this case, the adjunction arrow $\mathbf{1}_{\mathcal{D}}\to j_*j^*$ (resp., $j_!j^*\to\mathbf{1}_{\mathcal{D}}$) implies triangles*

$$U\to X\to j_*j^*X\to U[1]$$
$$(\text{resp., } j_!j^*X\to X\to V\to X[1])$$

with $U\in\operatorname{Ker}j^*$, $j_*j^*X\in\operatorname{Im}j_*$ (resp., $j_!j^*X\in\operatorname{Im}j_!$, $V\in\operatorname{Ker}j^*$) for all $X\in\mathcal{D}$.

(2) *If $\{\mathcal{D},\mathcal{D}'';j_!,j^*,j_*\}$ is a bilocalization of $\mathcal{D}$, then the canonical embedding $i_*:\operatorname{Ker}j^*\to\mathcal{D}$ has a right adjoint $i^!:\mathcal{D}\to\operatorname{Ker}j^*$ and a left adjoint $i^*:\mathcal{D}\to\operatorname{Ker}j^*$ such that $\{\operatorname{Ker}j^*,\mathcal{D},\mathcal{D}'';i^*,i_*,i^!,j_!,j^*,j_*\}$ is a recollement in the sense of* [1].

(3) *If $\{\mathcal{D}',\mathcal{D},\mathcal{D}'';i^*,i_*,i^!,j_!,j^*,j_*\}$ is a recollement, then $\{\mathcal{D},\mathcal{D}'';j_!,j^*,j_*\}$ is a bilocalization of $\mathcal{D}$.*

Proposition 2.4 ([1]). *Let $\{\mathcal{D}',\mathcal{D},\mathcal{D}'';i^*,i_*,i^!,j_!,j^*,j_*\}$ be a recollement, then $(\operatorname{Im}i_*,\operatorname{Im}j_*)$ and $(\operatorname{Im}j_!,\operatorname{Im}i_*)$ are stable t-structures in $\mathcal{D}$. Moreover, the adjunction arrows $\alpha:i_*i^!\to\mathbf{1}_{\mathcal{D}}$, $\beta:\mathbf{1}_{\mathcal{D}}\to j_*j^*$, $\gamma:j_!j^*\to\mathbf{1}_{\mathcal{D}}$, $\delta:\mathbf{1}_{\mathcal{D}}\to i_*i^*$ imply triangles in $\mathcal{D}$:*

$$i_*i^!X\xrightarrow{\alpha_X}X\xrightarrow{\beta_X}j_*j^*X\to i_*i^!X[1],$$
$$j_!j^*X\xrightarrow{\gamma_X}X\xrightarrow{\delta_X}i_*i^*X\to j_!j^*X[1],$$

for any $X\in\mathcal{D}$.

By Definition 2.1, we have the following properties.

Corollary 2.5. *Under the condition of Proposition 2.4, the following hold for $X\in\mathcal{D}$.*

(1) *$i_*i^!X\cong X$ (resp., $X\cong j_*j^*X$) in $\mathcal{D}$ if and only if α_X (resp., β_X) is an isomorphism.*

(2) *$j_!j^*X\cong X$ (resp., $X\cong i_*i^*X$) in $\mathcal{D}$ if and only if γ_X (resp., δ_X) is an isomorphism.*

For $X\in\mathsf{Mod}\,C^\circ\otimes A$, $Q\in\mathsf{Mod}\,B^\circ\otimes A$, let

$$\tau_Q(X):X\otimes_A\operatorname{Hom}_A(Q,A)\to\operatorname{Hom}_A(Q,X)$$

be the morphism in $\mathsf{Mod}\,C^\circ\otimes B$ defined by $(x\otimes f\mapsto(q\mapsto xf(q)))$ for $x\in X,q\in Q$, $f\in\operatorname{Hom}_A(Q,A)$. We have the following functorial isomorphism of derived functors.

Lemma 2.6. *Let k be a commutative ring, A, B, C k-projective k-algebras, ${}_BV^{\bullet}_A \in \mathsf{D}(B^{\circ}\otimes A)$ with $Res_A V^{\bullet} \in \mathsf{D}(A)_{\mathrm{perf}}$, and $V^{\star\bullet} = \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(V^{\bullet}, A) \in \mathsf{D}(A^{\circ}\otimes B)$. Then we have the ($\partial$-functorial) isomorphism:*

$$\tau_V : - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(V^{\bullet}, -)$$

as derived functors $\mathsf{D}(C^{\circ}\otimes A) \to \mathsf{D}(C^{\circ}\otimes B)$.

Proof. It is easy to see that we have a ∂-functorial morphism of derived functors $\mathsf{D}(C^{\circ}\otimes A) \to \mathsf{D}(C^{\circ}\otimes B)$:

$$\tau_V : - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \to \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(V^{\bullet}, -).$$

Let $P^{\bullet} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ which has a quasi-isomorphism $P^{\bullet} \to Res_A V^{\bullet}$. Then we have a ∂-functorial isomorphism of ∂-functors $\mathsf{D}(C^{\circ}\otimes A) \to \mathsf{D}(C^{\circ})$

$$\tau_P : - \overset{\bullet}{\otimes}_A \mathrm{Hom}^{\bullet}_A(P^{\bullet}, A) \xrightarrow{\sim} \mathrm{Hom}^{\bullet}_A(P^{\bullet}, -).$$

Since $Res_{C^{\circ}} \circ \tau_V \cong \tau_P$ and $\mathrm{H}^{\bullet}(\tau_P)$ is an isomorphism, τ_V is a ∂-functorial isomorphism. □

Concerning adjoints of the derived functor $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}$, by direct calculation we have the following properties.

Lemma 2.7. *Let k be a commutative ring, A, B, C k-projective k-algebras, ${}_BV^{\bullet}_A \in \mathsf{D}(B^{\circ}\otimes A)$ with $Res_A V^{\bullet} \in \mathsf{D}(A)_{\mathrm{perf}}$, and ${}_AV^{\star\bullet}_B = \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(V^{\bullet}, A) \in \mathsf{D}(A^{\circ}\otimes B)$. Then the following hold.*

1. *τ_V induces the adjoint isomorphism:*

   $$\Phi : \mathrm{Hom}_{\mathsf{D}(C^{\circ}\otimes B)}(-, ? \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(C^{\circ}\otimes A)}(- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet}, ?).$$

   Therefore, we get the morphism $\varepsilon_V : V^{\star\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet} \to A$ in $\mathsf{D}(A^{\mathrm{e}})$ (resp., $\vartheta_V : B \to V^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}$ in $\mathsf{D}(B^{\mathrm{e}})$) from the adjunction arrow of $A \in \mathsf{D}(A^{\mathrm{e}})$ (resp., $B \in \mathsf{D}(B^{\mathrm{e}})$).
2. *In the adjoint isomorphism of 1, the adjunction arrow $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet} \to \mathbf{1}_{\mathsf{D}(C^{\circ}\otimes A)}$ (resp., $\mathbf{1}_{\mathsf{D}(C^{\circ}\otimes B)} \to - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}$) is isomorphic to $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \varepsilon_V$ (resp., $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \vartheta_V$).*
3. *In the adjoint isomorphism:*

   $$\mathrm{Hom}_{\mathsf{D}(C^{\circ}\otimes A)}(-, \boldsymbol{R}\mathrm{Hom}^{\bullet}_B(V^{\star\bullet}, ?)) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(C^{\circ}\otimes B)}(- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}, ?),$$

   the adjunction arrow $\mathbf{1}_{\mathsf{D}(C^{\circ}\otimes A)} \to \boldsymbol{R}\mathrm{Hom}^{\bullet}_B(V^{\star\bullet}, - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet})$ (resp., $\boldsymbol{R}\mathrm{Hom}^{\bullet}_B(V^{\star\bullet}, -) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \to \mathbf{1}_{\mathsf{D}(C^{\circ}\otimes B)}$) is isomorphic to $\boldsymbol{R}\mathrm{Hom}^{\bullet}_A(\varepsilon_V, -)$ (resp., $\boldsymbol{R}\mathrm{Hom}^{\bullet}_B(\vartheta_V, -)$).

Let A, B be k-projective algebras over a commutative ring k. For a partial tilting complex $P^{\bullet} \in \mathsf{D}(A)$ with $B \cong \mathrm{End}_{\mathsf{D}(A)}(P^{\bullet})$, let ${}_BV^{\bullet}_A$ be the associated bimodule complex of $P^\bullet$. By Lemma 2.6, we can take

$$j_{V!} = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet : \mathsf{D}(B) \to \mathsf{D}(A),$$
$$j_V^* = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \cong \boldsymbol{R}\operatorname{Hom}^\bullet_A(V^\bullet, -) : \mathsf{D}(A) \to \mathsf{D}(B),$$
$$j_{V*} = \boldsymbol{R}\operatorname{Hom}^\bullet_B(V^{\star\bullet}, -) : \mathsf{D}(B) \to \mathsf{D}(A).$$

By Lemma 2.7, we get the triangle ξ_V in $\mathsf{D}(A^{\mathrm{e}})$:

$$V^{\star\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet \xrightarrow{\varepsilon_V} A \xrightarrow{\eta_V} \Delta^\bullet_A(V^\bullet) \to V^{\star\bullet}\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet[1].$$

Let $\mathcal{K}_P$ be the full subcategory of $\mathsf{D}(A)$ consisting of complexes $X^\bullet$ such that $\operatorname{Hom}_{\mathsf{D}(A)}(P^\bullet, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$.

Theorem 2.8. *Let A, B be k-projective algebras over a commutative ring k, $P^\bullet \in \mathsf{D}(A)$ a partial tilting complex with $B \cong \operatorname{End}_{\mathsf{D}(A)}(P^\bullet)$, and let ${}_BV^\bullet_A$ be the associated bimodule complex of $P^\bullet$. Take*

$$i_V^* = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^\bullet_A(V^\bullet) : \mathsf{D}(A) \to \mathcal{K}_P, \qquad j_{V!} = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet : \mathsf{D}(B) \to \mathsf{D}(A),$$
$$i_{V*} = \text{ the embedding } : \mathcal{K}_P \to \mathsf{D}(A), \qquad j_V^* = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} : \mathsf{D}(A) \to \mathsf{D}(B),$$
$$i_V^! = \boldsymbol{R}\operatorname{Hom}^\bullet_A(\Delta^\bullet_A(V^\bullet), -) : \mathsf{D}(A) \to \mathcal{K}_P, \qquad j_{V*} = \boldsymbol{R}\operatorname{Hom}^\bullet_B(V^{\star\bullet}, -) : \mathsf{D}(B) \to \mathsf{D}(A),$$

then $\{\mathcal{K}_P, \mathsf{D}(A), \mathsf{D}(B); i_V^*, i_{V*}, i_V^!, j_{V!}, j_V^*, j_{V*}\}$:

$$\mathcal{K}_P \ \overset{\leftarrow}{\rightleftarrows}\ \mathsf{D}(A)\ \overset{\leftarrow}{\rightleftarrows}\ \mathsf{D}(B)$$

is a recollement.

Proof. Since it is easy to see that $\tau_V(V^\bullet) \circ \vartheta_V$ is the left multiplication morphism $B \to \boldsymbol{R}\operatorname{Hom}^\bullet_A(V^\bullet, V^\bullet)$, by the remark of Definition 1.5, $\vartheta_V : B \to V^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}$ is an isomorphism in $\mathsf{D}(B^{\mathrm{e}})$. By Lemma 2.7, $\{\mathsf{D}(A), \mathsf{D}(B); j_{V!}, j_V^*, j_{V*}\}$ is a bilocalization. By Proposition 2.3, there exist $i_V^* : \mathsf{D}(A) \to \mathcal{K}_P$, $i_{V*} =$ the embedding $: \mathcal{K}_P \to \mathsf{D}(A)$, $i_V^! : \mathsf{D}(A) \to \mathcal{K}_P$ such that $\{\mathcal{K}_P, \mathsf{D}(A), \mathsf{D}(B); i_V^*, i_{V*}, i_V^!, j_{V!}, j_V^*, j_{V*}\}$ is a recollement. For $X^\bullet \in \mathsf{D}(A)$, by Lemma 2.7, $X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \varepsilon_V$ is isomorphic to the adjunction arrow $j_{V!}j_V^*(X^\bullet) \to X^\bullet$. Then $X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \eta_V$ is isomorphic to the adjunction arrow $X^\bullet \to i_{V*}i_V^*(X^\bullet)$, and hence we can take $i_V^* = -\overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^\bullet_A(V^\bullet)$ by Propositions 2.2, 2.4. Similarly, we can take $i_V^! = \boldsymbol{R}\operatorname{Hom}^\bullet_A(\Delta^\bullet_A(V^\bullet), -)$. □

In general, the above $\Delta^\bullet_A(V^\bullet)$ and $\Delta^\bullet_A(e)$ in Proposition 2.17 are unbounded complexes. Then, by the following corollary we have unbounded complexes which are compact objects in $\mathcal{K}_P$ and in $\mathsf{D}_{A/AeA}(A)$. This shows that recollements of Theorem 2.8 and Proposition 2.17 are out of localizations of triangulated categories which Neeman treated in [12].

Corollary 2.9. *Under the condition Theorem 2.8, the following hold.*

(1) *$\mathcal{K}_P$ is closed under coproducts in $\mathsf{D}(A)$.*

(2) *For any $X^\bullet \in \mathsf{D}(A)_{\mathrm{perf}}$, $X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^\bullet_A(V^\bullet)$ is a compact object in $\mathcal{K}_P$.*

Proof. 1. Since $P^\bullet$ is a compact object in $\mathsf{D}(A)$, it is trivial.

2. Since we have an isomorphism:

$$\operatorname{Hom}_{\mathsf{D}(A)}(i_V^* X^\bullet, Y^\bullet) \cong \operatorname{Hom}_{\mathsf{D}(A)}(X^\bullet, Y^\bullet)$$

for any $Y^\bullet \in \mathcal{K}_P$, we have the statement. □

Corollary 2.10. *Let A, B be k-projective algebras over a commutative ring k, $P^\bullet \in \mathsf{D}(A)$ a partial tilting complex with $B \cong \mathrm{End}_{\mathsf{D}(A)}(P^\bullet)$, and let ${}_BV^\bullet_A$ be the associated bimodule complex of $P^\bullet$. Then the following hold.*

1. $\Delta^\bullet_A(V^\bullet) \cong \Delta^\bullet_A(V^\bullet) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^\bullet_A(V^\bullet)$ *in* $\mathsf{D}(A^{\mathrm{e}})$.
2. $\boldsymbol{R}\mathrm{Hom}^\bullet_A(\Delta^\bullet_A(V^\bullet), \Delta^\bullet_A(V^\bullet)) \cong \Delta^\bullet_A(V^\bullet)$ *in* $\mathsf{D}(A^{\mathrm{e}})$.

Proof. Since $\Delta^\bullet_A(V^\bullet) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet}[n] \cong j_V^* i_{V*} i_V^*(A[n]) = 0$ for all n, $\Delta^\bullet_A(V^\bullet) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \eta_V$ is an isomorphism in $\mathsf{D}(A^{\mathrm{e}})$. Similarly, since

$$\begin{aligned}\boldsymbol{R}\mathrm{Hom}^\bullet_A(V^{\star\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet, \Delta^\bullet_A(V^\bullet))[n] &\cong \boldsymbol{R}\mathrm{Hom}^\bullet_B(V^{\star\bullet}, \Delta^\bullet_A(V^\bullet) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet})[n] \\ &= 0\end{aligned}$$

for all n, $\boldsymbol{R}\mathrm{Hom}^\bullet_A(\eta_V, \Delta^\bullet_A(V^\bullet))$ is an isomorphism in $\mathsf{D}(A^{\mathrm{e}})$. □

Lemma 2.11. *Let $\mathcal{D}$ be a triangulated category with coproducts. Then the following hold.*

1. *For morphisms of triangles in $\mathcal{D}$ ($n \geq 1$):*

$$\begin{array}{ccccccc} L_n & \longrightarrow & M_n & \longrightarrow & N_n & \longrightarrow & L_n[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_{n+1} & \longrightarrow & M_{n+1} & \longrightarrow & N_{n+1} & \longrightarrow & L_{n+1}[1], \end{array}$$

there exists a triangle $\coprod L_n \to \coprod L_n \to L \to \coprod L_n[1]$ such that we have the following triangle in $\mathcal{D}$:

$$L \to \underrightarrow{\mathsf{hocolim}} M_n \to \underrightarrow{\mathsf{hocolim}} N_n \to L[1].$$

2. *For a family of triangles in $\mathcal{D}$: $C_n \to X_{n-1} \to X_n \to C_n[1]$ ($n \geq 1$), with $X_0 = X$, there exists a family of triangles in $\mathcal{D}$:*

$$C_n[-1] \to Y_{n-1} \to Y_n \to C_n \ (n \geq 1),$$

with $Y_0 = 0$, such that we have the following triangle in $\mathcal{D}$:

$$Y \to X \to \underrightarrow{\mathsf{hocolim}} X_n \to Y[1],$$

where $\coprod Y_n \to \coprod Y_n \to Y \to \coprod Y_n[1]$ is a triangle in $\mathcal{D}$.

Proof. 1. By the assumption, we have a commutative diagram:

$$\begin{array}{ccccccc} \coprod L_n & \longrightarrow & \coprod M_n & \longrightarrow & \coprod N_n & \longrightarrow & \coprod L_n[1] \\ & & \big\downarrow{\scriptstyle 1-\mathrm{shift}} & & \big\downarrow{\scriptstyle 1-\mathrm{shift}} & & \\ \coprod L_n & \longrightarrow & \coprod M_n & \longrightarrow & \coprod N_n & \longrightarrow & \coprod L_n[1] \end{array}$$

According to [1] 9 lemma, we have the statement.

2. By the octahedral axiom, we have a commutative diagram:

$$\begin{array}{ccccccc}
 & & & & C_n & = & C_n \\
 & & & & \downarrow & & \downarrow \\
Y_{n-1} & \longrightarrow & X & \longrightarrow & X_{n-1} & \longrightarrow & Y_{n-1}[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
Y_n & \longrightarrow & X & \longrightarrow & X_n & \longrightarrow & Y_n[1] \\
 & & & & \downarrow & & \downarrow \\
 & & & & C_n[1] & = & C_n[1],
\end{array}$$

where all lines are triangles in $\mathcal{D}$. By 1, we have the statement. □

For an object M in an additive category $\mathcal{B}$, we denote by $\mathsf{Add}\, M$ (resp., $\mathsf{add}\, M$) the full subcategory of $\mathcal{B}$ consisting of objects which are isomorphic to summands of coproducts (resp., finite coproducts) of copies of M.

Definition 2.12. *Let A be a k-projective algebra over a commutative ring k, and $P^\cdot \in \mathsf{D}(A)$ a partial tilting complex. For $X^\cdot \in \mathsf{D}^-(A)$, there exists an integer r such that $\mathrm{Hom}_{\mathsf{D}(A)}(P^\cdot, X^\cdot[r+i]) = 0$ for all $i > 0$. Let $X^\cdot_0 = X^\cdot$. For $n \geq 1$, by induction we construct a triangle:*

$$P^\cdot_n[n-r-1] \xrightarrow{g_n} X^\cdot_{n-1} \xrightarrow{h_n} X^\cdot_n \to P^\cdot_n[n-r]$$

as follows. If $\mathrm{Hom}_{\mathsf{D}(A)}(P^\cdot, X^\cdot_{n-1}[r-n+1]) = 0$, then we set $P^\cdot_n = 0$. Otherwise, we take $P^\cdot_n \in \mathsf{Add}\, P^\cdot$ and a morphism $g'_n : P^\cdot_n \to X^\cdot_{n-1}[r-n+1]$ such that $\mathrm{Hom}_{\mathsf{D}(A)}(P^\cdot, g'_n)$ is an epimorphism, and let $g_n = g'_n[n-r-1]$. By Lemma 2.11, we have triangles:

$$P^\cdot_n[n-r-2] \to Y^\cdot_{n-1} \to Y^\cdot_n \to P^\cdot_n[n-r-1]$$

and $Y^\cdot_0 = 0$. Then we define $\nabla^\cdot_\infty(P^\cdot, X^\cdot)$ and $\Delta^\cdot_\infty(P^\cdot, X^\cdot)$ to be the complex Y of Lemma 2.11 (2) and $\underrightarrow{\mathsf{hocolim}} X^\cdot_n$, respectively. Moreover, we have a triangle:

$$\nabla^\cdot_\infty(P^\cdot, X^\cdot) \to X^\cdot \to \Delta^\cdot_\infty(P^\cdot, X^\cdot) \to \nabla^\cdot_\infty(P^\cdot, X^\cdot)[1].$$

Lemma 2.13. *Let A, B be k-projective algebras over a commutative ring k, $P^\cdot \in \mathsf{D}(A)$ a partial tilting complex with $B \cong \mathrm{End}_{\mathsf{D}(A)}(P^\cdot)$, and ${}_BV^\cdot_A$ the associated bimodule complex of $P^\cdot$. For $X^\cdot \in \mathsf{D}^-(A)$, we have an isomorphism of triangles in $\mathsf{D}(A)$:*

$$\begin{array}{ccccccc}
j_{V!}j_V^*X^\cdot & \longrightarrow & X^\cdot & \longrightarrow & i_{V*}i_V^*X^\cdot & \longrightarrow & j_{V!}j_V^*X^\cdot[1] \\
\downarrow\wr & & \| & & \downarrow\wr & & \downarrow\wr \\
\nabla^\cdot_\infty(P^\cdot, X^\cdot) & \longrightarrow & X^\cdot & \longrightarrow & \Delta^\cdot_\infty(P^\cdot, X^\cdot) & \longrightarrow & \nabla^\cdot_\infty(P^\cdot, X^\cdot)[1].
\end{array}$$

Proof. By the construction, we have $\mathrm{Hom}_{\mathsf{D}(A)}(P^\cdot, \Delta^\cdot_\infty(P^\cdot, X^\cdot)[i]) = 0$ for all i, and then $\Delta^\cdot_\infty(P^\cdot, X^\cdot) \in \mathrm{Im}\, i_{V*}$ (see Lemma 4.5). Since $j_{V!}$ is fully faithful and $P^\cdot \in \mathrm{Im}\, j_{V!}$, it is easy to see $Y^\cdot_n \in \mathrm{Im}\, j_{V!}$. Then $\nabla^\cdot_\infty(P^\cdot, X^\cdot) \in \mathrm{Im}\, j_{V!}$, because $j_{V!}$ commutes with coproducts. By Proposition 2.2, we complete the proof. □

Definition 2.14. *Let A be a k-projective algebra over a commutative ring k, and $P^\bullet \in \mathsf{D}(A)$ a partial tilting complex. Given $X^\bullet \in \mathsf{D}(A)$, for $n \geq 0$, we have a triangle:*

$$\nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) \to \sigma_{\leq n}X^\bullet \to \Delta^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) \to \nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet)[1].$$

According to Lemma 2.13 and Proposition 2.2, for $n \geq 0$ we have a morphism of triangles:

$$\begin{array}{ccccccc} \nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) & \to & \sigma_{\leq n}X^\bullet & \to & \Delta^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) & \to & \nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n+1}X^\bullet) & \to & \sigma_{\leq n+1}X^\bullet & \to & \Delta^\bullet_\infty(P^\bullet, \sigma_{\leq n+1}X^\bullet) & \to & \nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n+1}X^\bullet)[1]. \end{array}$$

Then we define $\nabla^\bullet_\infty(P^\bullet, X^\bullet)$ and $\Delta^\bullet_\infty(P^\bullet, X^\bullet)$ to be the complex L of Lemma 2.11 (1) and $\underrightarrow{\mathsf{hocolim}}\Delta^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet)$, respectively. Moreover, we have a triangle:

$$\nabla^\bullet_\infty(P^\bullet, X^\bullet) \to X^\bullet \to \Delta^\bullet_\infty(P^\bullet, X^\bullet) \to \nabla^\bullet_\infty(P^\bullet, X^\bullet)[1],$$

because $X^\bullet \cong \underrightarrow{\mathsf{hocolim}}\ \sigma_{\leq n}X^\bullet$.

Proposition 2.15. *Let A, B be k-projective algebras over a commutative ring k, $P^\bullet \in \mathsf{D}(A)$ a partial tilting complex with $B \cong \operatorname{End}_{\mathsf{D}(A)}(P^\bullet)$, and ${}_BV^\bullet_A$ the associated bimodule complex of $P^\bullet$. For $X^\bullet \in \mathsf{D}(A)$, we have an isomorphism of triangles in $\mathsf{D}(A)$:*

$$\begin{array}{ccccccc} j_{V!}j_V^*X^\bullet & \longrightarrow & X^\bullet & \longrightarrow & i_{V*}i_V^*X^\bullet & \longrightarrow & j_{V!}j_V^*X^\bullet[1] \\ \downarrow\wr & & \| & & \downarrow\wr & & \downarrow\wr \\ \nabla^\bullet_\infty(P^\bullet, X^\bullet) & \longrightarrow & X^\bullet & \longrightarrow & \Delta^\bullet_\infty(P^\bullet, X^\bullet) & \longrightarrow & \nabla^\bullet_\infty(P^\bullet, X^\bullet)[1]. \end{array}$$

Proof. By Lemma 2.13, $\nabla^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) \in \operatorname{Im} j_{V!}$ and $\Delta^\bullet_\infty(P^\bullet, \sigma_{\leq n}X^\bullet) \in \operatorname{Im} i_{V*}$. Since $P^\bullet$ is a perfect complex, $\operatorname{Hom}_{\mathsf{D}(A)}(P^\bullet, -)$ commutes with coproducts. Then we have $\Delta^\bullet_\infty(P^\bullet, X^\bullet) \in \operatorname{Im} i_{V*}$. We have also $\nabla^\bullet_\infty(P^\bullet, X^\bullet) \in \operatorname{Im} j_{V!}$, because $j_{V!}$ is fully faithful and commutes with coproducts. By Proposition 2.2, we complete the proof. □

Corollary 2.16. *Let A, B be k-projective algebras over a commutative ring k, $P^\bullet \in \mathsf{D}(A)$ a partial tilting complex with $B \cong \operatorname{End}_{\mathsf{D}(A)}(P^\bullet)$, and ${}_BV^\bullet_A$ the associated bimodule complex of $P^\bullet$. For $X^\bullet \in \mathsf{D}(A)$, we have isomorphisms in $\mathsf{D}(A)$:*

$$X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A V^{\star\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B V^\bullet \cong \nabla^\bullet_\infty(P^\bullet, X^\bullet),$$

$$X^\bullet \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^\bullet_A(V^\bullet) \cong \Delta^\bullet_\infty(P^\bullet, X^\bullet).$$

Proof. By Theorem 2.8 and Proposition 2.15, we complete the proof. □

For an idempotent e of a ring A, by $\operatorname{Hom}_A(eA, A) \cong Ae$, we have

$$\begin{aligned} j^e_{A!} &= - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA : \mathsf{D}(eAe) \to \mathsf{D}(A), \\ j^{e*}_A &= - \otimes_A Ae \cong \operatorname{Hom}_A(eA, -) : \mathsf{D}(A) \to \mathsf{D}(eAe), \\ j^e_{A*} &= \boldsymbol{R}\operatorname{Hom}^\bullet_{eAe}(Ae, -) : \mathsf{D}(eAe) \to \mathsf{D}(A). \end{aligned}$$

And we also get the triangle ξ_e in $\mathsf{D}(A^{\mathrm{e}})$:

$$Ae \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA \xrightarrow{\varepsilon_e} A \xrightarrow{\eta_e} \Delta^\bullet_A(e) \to Ae \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA[1].$$

Throughout this paper, we identify $\mathsf{Mod}\, A/AeA$ with the full subcategory of $\mathsf{Mod}\, A$ consisting of A-modules M such that $\operatorname{Hom}_A(eA, M) = 0$. We denote by $\mathsf{D}^*_{A/AeA}(A)$

the full subcategory of $\mathsf{D}^*(A)$ consisting of complexes whose cohomologies are in $\mathsf{Mod}\, A/AeA$, where $* =$ nothing, $+, -, \mathrm{b}$. According to Theorem 2.8, we have the following.

Proposition 2.17. *Let A be a k-projective algebra over a commutative ring k, e an idempotent of A, and let*

$$\begin{array}{ll} i_A^{e*} = - \dot{\otimes}^{\boldsymbol{L}}_A \Delta^{\cdot}_A(e) : \mathsf{D}(A) \to \mathsf{D}_{A/AeA}(A), & j_{A!}^{e} = - \dot{\otimes}^{\boldsymbol{L}}_{eAe} eA : \mathsf{D}(eAe) \to \mathsf{D}(A), \\ i_{A*}^{e} = \text{ the embedding } : \mathsf{D}_{A/AeA}(A) \to \mathsf{D}(A), & j_A^{e*} = - \otimes_A Ae : \mathsf{D}(A) \to \mathsf{D}(eAe), \\ i_A^{e!} = \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_A(e), -) : \mathsf{D}(A) \to \mathsf{D}_{A/AeA}(A), & j_{A*}^{e} = \boldsymbol{R}\operatorname{Hom}^{\cdot}_{eAe}(Ae, -) : \mathsf{D}(eAe) \to \mathsf{D}(A). \end{array}$$

Then $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe); i_A^{e*}, i_{A*}^{e}, i_A^{e!}, j_{A!}^{e}, j_A^{e*}, j_{A*}^{e}\}$ is a recollement.

Remark 2.18. *According to Proposition 1.1 and Lemma 2.7, it is easy to see that $\{\mathsf{D}_{C^\circ \otimes A/AeA}(C^\circ \otimes A), \mathsf{D}(C^\circ \otimes A), \mathsf{D}(C^\circ \otimes eAe); i_A^{e*}, i_{A*}^{e}, i_A^{e!}, j_{A!}^{e}, j_A^{e*}, j_{A*}^{e}\}$ is also a recollement for any k-projective k-algebra C.*

Corollary 2.19. *Let A be a k-projective algebra over a commutative ring k, and e an idempotent of A, then the following hold.*

1. $\Delta^{\cdot}_A(e) \dot{\otimes}^{\boldsymbol{L}}_A \Delta^{\cdot}_A(e) \cong \Delta^{\cdot}_A(e)$ *in* $\mathsf{D}(A^{\mathrm{e}})$
2. $\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_A(e), \Delta^{\cdot}_A(e)) \cong \Delta^{\cdot}_A(e)$ *in* $\mathsf{D}(A^{\mathrm{e}})$
3. *We have the following isomorphisms in* $\mathsf{Mod}\, A^{\mathrm{e}}$*:*
$$A/AeA \cong \operatorname{End}_{\mathsf{D}(A)}(\Delta^{\cdot}_A(e)) \cong \mathrm{H}^0(\Delta^{\cdot}_A(e)).$$
Moreover, the first isomorphism is a ring isomorphism.

Proof. 1, 2. By Corollary 2.10.
3. Applying $\operatorname{Hom}_{\mathsf{D}(A)}(-, \Delta^{\cdot}_A(e))$ to ξ_e, we have an isomorphism in $\mathsf{Mod}\, A^{\mathrm{e}}$:

$$\operatorname{Hom}_{\mathsf{D}(A)}(\Delta^{\cdot}_A(e), \Delta^{\cdot}_A(e)) \cong \operatorname{Hom}_{\mathsf{D}(A)}(A, \Delta^{\cdot}_A(e)),$$

because $\operatorname{Hom}_{\mathsf{D}(A)}(Ae \dot{\otimes}^{\boldsymbol{L}}_{eAe} eA, \Delta^{\cdot}_A(e)[n]) \cong \operatorname{Hom}_{\mathsf{D}(A)}(j_{A!}^{e} j_A^{e*}(A), i_{A*}^{e} i_A^{e!}(A)[n]) = 0$ for all $n \in \mathbb{Z}$ by Proposition 2.3, 1. Applying $\operatorname{Hom}_{\mathsf{D}(A)}(A, -)$ to ξ_e, we have an isomorphism between exact sequences in $\mathsf{Mod}\, A^{\mathrm{e}}$:

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathsf{D}(A)}(A, Ae \dot{\otimes}^{\boldsymbol{L}}_{eAe} eA) & \to & \operatorname{Hom}_{\mathsf{D}(A)}(A, A) & \to & \operatorname{Hom}_{\mathsf{D}(A)}(A, \Delta^{\cdot}_A(e)) \to 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ Ae \otimes_{eAe} eA & \longrightarrow & A & \longrightarrow & A/AeA \to 0. \end{array}$$

Consider the inverse of $\operatorname{Hom}_{\mathsf{D}(A)}(\Delta^{\cdot}_A(e), \Delta^{\cdot}_A(e)) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}(A)}(A, \Delta^{\cdot}_A(e))$, then it is easy to see that $\operatorname{Hom}_{\mathsf{D}(A)}(A, A) \to \operatorname{Hom}_{\mathsf{D}(A)}(A, \Delta^{\cdot}_A(e)) \to \operatorname{Hom}_{\mathsf{D}(A)}(\Delta^{\cdot}_A(e), \Delta^{\cdot}_A(e))$ is a ring morphism. □

Remark 2.20. *It is not hard to see that the above triangle ξ_e also play the same role in the left module version of Corollary 2.19. Then we have also*

1. $\boldsymbol{R}\operatorname{Hom}^{\cdot}_{A^\circ}(\Delta^{\cdot}_A(e), \Delta^{\cdot}_A(e)) \cong \Delta^{\cdot}_A(e)$ *in* $\mathsf{D}(A^{\mathrm{e}})$
2. *We have a ring isomorphism* $(A/AeA)^\circ \cong \operatorname{End}_{\mathsf{D}(A^\circ)}(\Delta^{\cdot}_A(e))$.

3. Equivalences between Recollements

In this section, we study triangle equivalences between recollements induced by idempotents.

Definition 3.1. *Let $\{\mathcal{D}_n, \mathcal{D}''_n; j_{n*}, j^*_n\}$ (resp., $\{\mathcal{D}_n, \mathcal{D}''_n; j_{n!}, j^*_n, j_{n*}\}$) be a colocalization (resp., a bilocalization) of $\mathcal{D}_n$ ($n = 1, 2$). If there are triangle equivalences $F : \mathcal{D}_1 \to \mathcal{D}_2$, $F'' : \mathcal{D}''_1 \to \mathcal{D}''_2$ such that all squares are commutative up to (∂-functorial) isomorphism in the diagram:*

$$
\begin{array}{ccc}
\mathcal{D}_1 & \leftrightarrows & \mathcal{D}''_1 \\
F\downarrow & & \downarrow F'' \\
\mathcal{D}_2 & \leftrightarrows & \mathcal{D}''_2
\end{array}
\qquad \left(\textit{resp.,}\quad
\begin{array}{ccc}
\mathcal{D}_1 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}''_1 \\
F\downarrow & & \downarrow F'' \\
\mathcal{D}_2 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}''_2
\end{array}\right),
$$

then we say that a colocalization $\{\mathcal{D}_1, \mathcal{D}''_1; j_{n*}, j^*_1\}$ (resp., a bilocalization $\{\mathcal{D}_1, \mathcal{D}''_1; j_{1!}, j^*_1, j_{1*}\}$) is triangle equivalent to a colocalization $\{\mathcal{D}_2, \mathcal{D}''_2; j_{n*}, j^*_2\}$ (resp., a bilocalization $\{\mathcal{D}_2, \mathcal{D}''_2; j_{n!}, j^*_2, j_{2*}\}$).

For recollements $\{\mathcal{D}'_n, \mathcal{D}_n, \mathcal{D}''_n; i^*_n, i_{n*}, i^!_n, j_{n!}, j^*_n, j_{n*}\}$ ($n = 1, 2$), if there are triangle equivalences $F' : \mathcal{D}'_1 \to \mathcal{D}'_2$, $F : \mathcal{D}_1 \to \mathcal{D}_2$, $F'' : \mathcal{D}''_1 \to \mathcal{D}''_2$ such that all squares are commutative up to (∂-functorial) isomorphism in the diagram:

$$
\begin{array}{ccccc}
\mathcal{D}'_1 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}_1 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}''_1 \\
F'\downarrow & & F\downarrow & & \downarrow F'' \\
\mathcal{D}'_2 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}_2 & \overset{\leftarrow}{\rightleftarrows} & \mathcal{D}''_2,
\end{array}
$$

then we say that a recollement $\{\mathcal{D}'_1, \mathcal{D}_1, \mathcal{D}''_1; i^*_1, i_{1*}, i^!_1, j_{1!}, j^*_1, j_{1*}\}$ is triangle equivalent to a recollement $\{\mathcal{D}'_2, \mathcal{D}_2, \mathcal{D}''_2; i^*_2, i_{2*}, i^!_2, j_{2!}, j^*_2, j_{2*}\}$.

We simply write a localization $\{\mathcal{D}, \mathcal{D}''\}$, etc. for a localization $\{\mathcal{D}, \mathcal{D}''; j^*, j_*\}$, etc. when we don't confuse them. Parshall and Scott showed the following.

Proposition 3.2 ([14])**.** *Let $\{\mathcal{D}'_n, \mathcal{D}_n, \mathcal{D}''_n\}$ be recollements ($n = 1, 2$). If triangle equivalences $F : \mathcal{D}_1 \to \mathcal{D}_2$, $F'' : \mathcal{D}''_1 \to \mathcal{D}''_2$ induce that a bilocalization $\{\mathcal{D}_1, \mathcal{D}''_1\}$ is triangle equivalent to a bilocalization $\{\mathcal{D}_2, \mathcal{D}''_2\}$, then there exists a unique triangle equivalence $F' : \mathcal{D}'_1 \to \mathcal{D}'_2$ up to isomorphism such that F', F, F'' induce that a recollement $\{\mathcal{D}'_1, \mathcal{D}_1, \mathcal{D}''_1\}$ is triangle equivalent to a recollement $\{\mathcal{D}'_2, \mathcal{D}_2, \mathcal{D}''_2\}$.*

Lemma 3.3. *Let A be a k-projective algebra over a commutative ring k, and e an idempotent of A. For $X^\cdot \in \mathsf{D}(A)_{\mathrm{perf}}$, the following are equivalent.*

(1) *$X^\cdot \cong P^\cdot$ in $\mathsf{D}(A)$ for some $P^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{add}\, eA)$.*
(2) *$j^e_{A!} j^{e*}_A(X^\cdot) \cong X^\cdot$ in $\mathsf{D}(A)$.*
(3) *γ_X is an isomorphism, where $\gamma : j^e_{A!} j^{e*}_A \to \mathbf{1}_{\mathsf{D}(A)}$ is the adjunction arrow.*

Proof. 1 $\Rightarrow$ 2. Since $j^e_{A!} j^{e*}_A(P) \cong P$ in $\mathsf{Mod}\, A$ for any $P \in \mathsf{add}\, eA$, it is trivial.
2 $\Leftrightarrow$ 3. By Corollary 2.5.

3 ⇒ 1. Let $\{Y_i^\cdot\}_{i\in I}$ be a family of complexes of $\mathsf{D}(A)$. By Proposition 1.3, we have isomorphisms:

$$\begin{aligned}\coprod_{i\in I}\mathrm{Hom}_{\mathsf{D}(eAe)}(j_A^{e*}(X^\cdot), j_A^{e*}(Y_i^\cdot)) &\cong \coprod_{i\in I}\mathrm{Hom}_{\mathsf{D}(A)}(j_{A!}^{e}j_A^{e*}(X^\cdot), Y_i^\cdot)\\ &\cong \coprod_{i\in I}\mathrm{Hom}_{\mathsf{D}(A)}(X^\cdot, Y_i^\cdot)\\ &\cong \mathrm{Hom}_{\mathsf{D}(A)}(X^\cdot, \coprod_{i\in I}Y_i^\cdot)\\ &\cong \mathrm{Hom}_{\mathsf{D}(A)}(j_{A!}^{e}j_A^{e*}(X^\cdot), \coprod_{i\in I}Y_i^\cdot)\\ &\cong \mathrm{Hom}_{\mathsf{D}(eAe)}(j_A^{e*}(X^\cdot), j_A^{e*}(\coprod_{i\in I}Y_i^\cdot))\\ &\cong \mathrm{Hom}_{\mathsf{D}(eAe)}(j_A^{e*}(X^\cdot), \coprod_{i\in I}j_A^{e*}(Y_i^\cdot)).\end{aligned}$$

Since any complex $Z^\cdot$ of $\mathsf{D}(eAe)$ is isomorphic to $j_A^{e*}(Y^\cdot)$ for some $Y^\cdot \in \mathsf{D}(A)$, by Proposition 1.3 the above isomorphisms imply that $j_A^{e*}(X^\cdot)$ is a perfect complex of $\mathsf{D}(eAe)$. Therefore, $j_{A!}^{e}j_A^{e*}(X^\cdot)$ is isomorphic to $P^\cdot$ for some $P^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{add}\, eA)$. □

Lemma 3.4. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For $X^\cdot, Y^\cdot \in \mathsf{D}(B^\circ \otimes A)$, we have an isomorphism in $\mathsf{D}((fBf)^{\mathrm{e}})$:*

$$fB \otimes_B \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, Y^\cdot) \otimes_B Bf \cong \boldsymbol{R}\mathrm{Hom}_A^\cdot(fX^\cdot, fY^\cdot).$$

Proof. First, by Proposition 1.1, 2, we have isomorphisms in $\mathsf{D}((fBf)^\circ \otimes B)$:

$$\begin{aligned}fB \otimes_B \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, Y^\cdot) &\cong \mathrm{Hom}_B(Bf, \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, Y^\cdot))\\ &\cong \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, \mathrm{Hom}_B(Bf, Y^\cdot))\\ &\cong \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, fY^\cdot).\end{aligned}$$

Then we have isomorphisms in $\mathsf{D}((fBf)^{\mathrm{e}})$:

$$\begin{aligned}fB \otimes_B \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, Y^\cdot) \otimes_B Bf &\cong \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, fY^\cdot) \otimes_B Bf\\ &\cong \mathrm{Hom}_B(fB, \boldsymbol{R}\mathrm{Hom}_A^\cdot(X^\cdot, fY^\cdot))\\ &\cong \boldsymbol{R}\mathrm{Hom}_A^\cdot(fX^\cdot, fY^\cdot).\end{aligned}$$

□

Theorem 3.5. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. Then the following are equivalent.*

1. *The colocalization $\{\mathsf{D}(A), \mathsf{D}(eAe); j_{A!}^{e}, j_A^{e*}\}$ is triangle equivalent to the colocalization $\{\mathsf{D}(B), \mathsf{D}(fBf); j_{B!}^{f}, j_B^{f*}\}$.*
2. *There is a tilting complex $P^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ such that $P^\cdot = P_1^\cdot \oplus P_2^\cdot$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ satisfying*
   (a) $B \cong \mathrm{End}_{\mathsf{D}(A)}(P^\cdot)$,
   (b) under the isomorphism of (a), $f \in B$ corresponds to the canonical morphism $P^\cdot \to P_1^\cdot \to P^\cdot \in \mathrm{End}_{\mathsf{D}(A)}(P^\cdot)$,
   (c) $P_1^\cdot \in \mathsf{K}^{\mathrm{b}}(\mathsf{add}\, eA)$, and $j_A^{e*}(P_1^\cdot)$ is a tilting complex for eAe.

(3) *The recollement* $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ *is triangle equivalent to the recollement* $\{\mathsf{D}_{B/BfB}(B), \mathsf{D}(B), \mathsf{D}(fBf)\}$.

Proof. $1 \Rightarrow 2$. Let $G : \mathsf{D}(B) \to \mathsf{D}(A)$, $G'' : \mathsf{D}(fBf) \to \mathsf{D}(eAe)$ be triangle equivalences such that

$$\begin{array}{ccc} \mathsf{D}(B) & \leftrightarrows & \mathsf{D}(fBf) \\ G\downarrow & & \downarrow G'' \\ \mathsf{D}(A) & \leftrightarrows & \mathsf{D}(eAe) \end{array}$$

is commutative up to isomorphism. Then $G(B)$ and $G''(fBf)$ are tilting complexes for A and for eAe with $B \cong \mathrm{End}_{\mathsf{D}(A)}(G(B))$, $fBf \cong \mathrm{End}_{\mathsf{D}(eAe)}(G''(B))$, respectively. Considering $G(B) = G(fB) \oplus G((1-f)B)$, by the above commutativity, we have isomorphisms:

$$\begin{aligned} G(fB) &\cong Gj^{f}_{B!}(fBf) \\ &\cong j^{e}_{A!}G''(fBf) \\ &\cong j^{e}_{A!}G''j^{f*}_{B}(fB) \\ &\cong j^{e}_{A!}j^{e*}_{A}G(fB), \end{aligned}$$

$$\begin{aligned} j^{e*}_{A}G(fB) &\cong G''j^{f*}_{B}(fB) \\ &\cong G''(fBf). \end{aligned}$$

By Lemma 3.3, $G(fB)$ is isomorphic to a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{add}\, eA)$, and $j^{e*}_{A}G(fB)$ is a tilting complex for eAe.

$2 \Rightarrow 3$. Let ${}_BT^{\bullet}_A$ be a two-sided tilting complex which is induced by $P^{\bullet}_A$. By the assumption, $Res_A(fT^{\bullet}) \cong P^{\bullet}_1$ in $\mathsf{D}(A)$. By Lemma 3.3, $\gamma_{fT} : j^{e}_{A!}j^{e*}_{A}(fT^{\bullet}) \xrightarrow{\sim} fT^{\bullet}$ is an isomorphism in $\mathsf{D}(A)$. By Remark 2.18, Proposition 1.1, 5, we have $fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA \cong fT^{\bullet}$ in $\mathsf{D}((fBf)^{\circ} \otimes A)$. By Proposition 1.8, Lemma 3.4, we have isomorphisms in $\mathsf{D}((fBf)^{\mathrm{e}})$:

$$\begin{aligned} fBf &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_{A}(fT^{\bullet}, fT^{\bullet}) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_{A}(fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA, fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_{A}(fT^{\bullet}e, fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eAe) \\ &\cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, fT^{\bullet}e). \end{aligned}$$

By taking cohomology, we have

$$fBf \cong \mathrm{Hom}_{\mathsf{D}(eAe)}(fT^{\bullet}e, fT^{\bullet}e).$$

By the assumption, $fT^{\bullet}e \cong j^{e*}_{A}(fT^{\bullet}) \cong j^{e*}_{A}(P^{\bullet}_1)$ is a tilting complex for eAe. Since it is easy to see the above isomorphism is induced by the left multiplication, by [17] Lemma 3.2, [9] Theorem, $fT^{\bullet}e$ is a two-sided tilting complex in $\mathsf{D}((fBf)^{\circ} \otimes eAe)$. Let

$$\begin{aligned} F &= \boldsymbol{R}\mathrm{Hom}^{\bullet}_{A}(T^{\bullet}, -) : \mathsf{D}(B^{\circ} \otimes A) \to \mathsf{D}(B^{\circ} \otimes B), \\ F'' &= \boldsymbol{R}\mathrm{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, -) : \mathsf{D}(B^{\circ} \otimes eAe) \to \mathsf{D}(B^{\circ} \otimes fBf), \\ G &= - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B} T^{\bullet} : \mathsf{D}(B^{\circ} \otimes B) \to \mathsf{D}(B^{\circ} \otimes A), \\ G'' &= - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e : \mathsf{D}(B^{\circ} \otimes eAe) \to \mathsf{D}(B^{\circ} \otimes fBf). \end{aligned}$$

Using the same symbols, consider a triangle equivalence between colocalizations $\{\mathsf{D}(B^{\circ} \otimes A), \mathsf{D}(B^{\circ} \otimes eAe); j^{e}_{A!}, j^{e*}_{A}\}$ and $\{\mathsf{D}(B^{\circ} \otimes B), \mathsf{D}(B^{\circ} \otimes fBf); j^{f}_{B!}, j^{f*}_{B}\}$. And we use the same symbols

$$F = \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, -) : \mathsf{D}(A) \to \mathsf{D}(B), \qquad F'' = \boldsymbol{R}\operatorname{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, -) : \mathsf{D}(eAe) \to \mathsf{D}(fBf),$$

$$G = - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B} T^{\bullet} : \mathsf{D}(B) \to \mathsf{D}(A), \qquad G'' = - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e : \mathsf{D}(eAe) \to \mathsf{D}(fBf).$$

For any $X^{\bullet} \in \mathsf{D}(B^{\circ} \otimes A)$ (resp., $X^{\bullet} \in \mathsf{D}(A)$), by Proposition 1.1, 3, we have isomorphisms in $\mathsf{D}(B^{\circ} \otimes fBf)$ (resp., $\mathsf{D}(fBf)$):

$$\begin{aligned} j_B^{f*}F(X^{\bullet}) &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(fB, \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, X^{\bullet})) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(fT^{\bullet}, X^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(j^e_{A!}j^{e*}_A(fT^{\bullet}), X^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_{eAe}(j^{e*}_A(fT^{\bullet}), j^{e*}_A(X^{\bullet})) \\ &\cong F''j^{e*}_A(X^{\bullet}). \end{aligned}$$

Since G, G'' are quasi-inverses of F, F'', respectively, for $B \in \mathsf{D}(B^{\circ} \otimes B)$ we have isomorphisms in $\mathsf{D}(B^{\circ} \otimes eAe)$:

$$\begin{aligned} T^{\bullet}e &\cong j^{e*}_A G(B) \\ &\cong G''j^{f*}_B(B) \\ &\cong Bf \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e. \end{aligned}$$

Therefore, for any $Y^{\bullet} \in \mathsf{D}(eAe)$, we have isomorphisms in $\mathsf{D}(B)$:

$$\begin{aligned} j^f_{B*}F''(Y^{\bullet}) &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_{fBf}(Bf, \boldsymbol{R}\operatorname{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, Y^{\bullet})) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(Bf \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e, Y^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(T^{\bullet}e, Y^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(j^{e*}_A(T^{\bullet}), Y^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(T^{\bullet}, j^e_{A*}(Y^{\bullet})) \\ &\cong Fj^e_{A*}(Y^{\bullet}). \end{aligned}$$

For any $Z^{\bullet} \in \mathsf{D}(fBf)$, we have isomorphisms in $\mathsf{D}(A)$:

$$\begin{aligned} j^e_{A!}G''(Z^{\bullet}) &= Z^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA \\ &\cong Z^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet} \\ &\cong Z^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fB \otimes_B T^{\bullet} \\ &\cong G''j^f_{B!}(Z^{\bullet}). \end{aligned}$$

Since F, F'' are quasi-inverses of G, G'', respectively, we have $j^f_{B!}F'' \cong Fj^e_{A!}$. By Proposition 3.2, we have the statement.

3 ⇒ 1. It is trivial. □

Definition 3.6. *Let A be a k-projective algebra over a commutative ring k, and e an idempotent of A. We call a tilting complex $P^{\bullet} \in \mathsf{K}^{\mathrm{b}}(\mathsf{projA})$ a recollement tilting complex related to an idempotent e of A if $P^{\bullet}$ satisfies the condition of Theorem 3.5, 2. In this case, we call an idempotent $f \in B$ an idempotent corresponding to e.*

We see the following symmetric properties of a two-sided tilting complex which is induced by a recollement tilting complex. We will call the following two-sided tilting complex a *two-sided recollement tilting complex* ${}_BT^{\bullet}_A$ *related to idempotents* $e \in A$, $f \in B$.

Corollary 3.7. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. Let ${}_BT^{\bullet}_A$ be a two-sided tilting complex such that*

(a) *$fT^{\bullet}e \in \mathsf{D}((fBf)^{\circ} \otimes eAe)$ is a two-sided tilting complex,*

(b) *$fT^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA \cong fT^{\bullet}$ in $\mathsf{D}((fBf)^{\circ} \otimes A)$.*

Then the following hold.

(1) *$Bf \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}e \cong T^{\bullet}e$ in $\mathsf{D}(B^{\circ} \otimes eAe)$.*
(2) *$eT^{\vee\bullet}f$ is the inverse of $fT^{\bullet}e$, where $T^{\vee\bullet}$ is the inverse of $T^{\bullet}$.*
(3) *$Ae \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eT^{\vee\bullet}f \cong T^{\vee\bullet}f$ in $\mathsf{D}(A^{\circ} \otimes fBf)$.*
(4) *$eT^{\vee\bullet}f \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fB \cong eT^{\vee\bullet}$ in $\mathsf{D}((eAe)^{\circ} \otimes B)$.*

Proof. Here we use the same symbols in the proof $2 \Rightarrow 3$ of Theorem 3.5. It is easy to see that F and F'' induce a triangle equivalence between bilocalizations $\{\mathsf{D}(B^{\circ}\otimes A), \mathsf{D}(B^{\circ}\otimes eAe); j^{e}_{A!}, j^{e*}_{A}, j^{e}_{A*}\}$ and $\{\mathsf{D}(B^{\circ}\otimes B), \mathsf{D}(B^{\circ}\otimes fBf); j^{f}_{B!}, j^{f*}_{B}, j^{f}_{B*}\}$. By the proof of Theorem 3.5, we get the statement 1, and $j^{f*}_{B}F \cong F''j^{e*}_{A}$, $j^{f}_{B!}F'' \cong Fj^{e}_{A!}$ and $j^{f}_{B*}F'' \cong Fj^{e}_{A*}$. Then we have isomorphisms $j^{f*}_{B}Fj^{e}_{A!} \cong F''j^{e*}_{A}j^{e}_{A!} \cong F''$. Since $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A} T^{\vee\bullet}_{B} \cong F$, we have isomorphisms $eT^{\vee\bullet}f \cong \boldsymbol{R}\mathrm{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, eAe)$ in $\mathsf{D}((eAe)^{\circ} \otimes fBf)$, and $- \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eT^{\vee\bullet}f \cong F''$. This means that $eT^{\vee\bullet}f$ is the inverse of a two-sided tilting complex $fT^{\bullet}e$. Similarly, $j^{f*}_{B}F \cong F''j^{e*}_{A}$ and $j^{f}_{B!}F'' \cong Fj^{e}_{A!}$ imply the statements 3 and 4, respectively. □

Corollary 3.8. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For a two-sided recollement tilting complex ${}_BT^{\bullet}_A$ related to idempotents e, f, we have an isomorphism between triangles $T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A} \xi_e$ and $\xi_f \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B} T^{\bullet}$ in $\mathsf{D}(B^{\circ} \otimes A)$:*

$$\begin{array}{ccccccc}
T^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA & \longrightarrow & T^{\bullet} & \longrightarrow & T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{A} \Delta^{\bullet}_{A}(e) & \longrightarrow & T^{\bullet}e \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{eAe} eA[1] \\
\downarrow\wr & & \| & & \downarrow\wr & & \downarrow\wr \\
Bf \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet} & \longrightarrow & T^{\bullet} & \longrightarrow & \Delta^{\bullet}_{B}(f) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{B} T^{\bullet} & \longrightarrow & Bf \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_{fBf} fT^{\bullet}[1].
\end{array}$$

Proof. According to Proposition 3.2, for the triangle equivalence between colocalizations in the proof of Corollary 3.7 there exists $F' : \mathsf{D}_{B^{\circ}\otimes B/BfB}(B^{\circ} \otimes B) \to \mathsf{D}_{B^{\circ}\otimes A/AeA}(B^{\circ} \otimes A)$ such that the recollement

$$\{\mathsf{D}_{B^{\circ}\otimes B/BfB}(B^{\circ} \otimes B), \mathsf{D}(B^{\circ} \otimes B), \mathsf{D}(B^{\circ} \otimes fBf); i^{f*}_{B}, i^{f}_{B*}, i^{f!}_{B}, j^{f}_{B!}, j^{f*}_{B}, j^{f}_{B*}\}$$

is triangle equivalent to the recollement

$$\{\mathsf{D}_{B^{\circ}\otimes A/AeA}(B^{\circ} \otimes A), \mathsf{D}(B^{\circ} \otimes A), \mathsf{D}(B^{\circ} \otimes eAe); i^{e*}_{A}, i^{e}_{A*}, i^{e!}_{A}, j^{e}_{A!}, j^{e*}_{A}, j^{e}_{A*}\}.$$

By Proposition 1.1, Lemma 2.7, the triangle $T^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_{A} \xi_e$ is isomorphic to the following triangle in $\mathsf{D}(B^{\circ} \otimes A)$:

$$j^e_{A!} j^{e*}_A (T^{\bullet}) \to T^{\bullet} \to i^e_{A*} i^{e*}_A (T^{\bullet}) \to j^e_{A!} j^{e*}_A (T^{\bullet})[1].$$

On the other hand, the triangle $\xi_f \dot{\otimes}^{\boldsymbol{L}}_{B} T^{\bullet}$ is isomorphic to the following triangle in $\mathsf{D}(B^{\circ} \otimes A)$:

$$F j^f_{B!} j^{f*}_B (B) \to F(B) \to F i^f_{B*} i^{f*}_B (B) \to F j^f_{B!} j^{f*}_B (B)[1].$$

Since $F(B) \cong T^{\bullet}$, $F j^f_{B!} j^{f*}_B (B) \cong j^e_{A!} F'' j^{f*}_B (B) \cong j^e_{A!} j^{e*}_A F(B)$, $F i^f_{B*} i^{f*}_B (B) \cong i^e_{A*} F' i^{f*}_B (B) \cong i^e_{A*} i^{e*}_A F(B)$, by Proposition 2.2, we complete the proof. □

Corollary 3.9. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For a two-sided recollement tilting complex ${}_B T^{\bullet}_A$ related to idempotents e, f, the following hold.*

(1) $T^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_{A} \Delta^{\bullet}_A(e) \cong \Delta^{\bullet}_B(f) \dot{\otimes}^{\boldsymbol{L}}_{B} T^{\bullet}$ *in* $\mathsf{D}(B^{\circ} \otimes A)$.

(2) $\Delta^{\bullet}_A(e) \dot{\otimes}^{\boldsymbol{L}}_{A} T^{\vee \bullet} \cong T^{\vee \bullet} \dot{\otimes}^{\boldsymbol{L}}_{B} \Delta^{\bullet}_B(f)$ *in* $\mathsf{D}(A^{\circ} \otimes B)$.

Proof. 1. By Corollary 3.8.

2. We have isomorphisms in $\mathsf{D}(A^{\circ} \otimes B)$:

$$\begin{aligned} \Delta^{\bullet}_A(e) \dot{\otimes}^{\boldsymbol{L}}_{A} T^{\vee \bullet} &\cong T^{\vee \bullet} \dot{\otimes}^{\boldsymbol{L}}_{B} T^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_{A} \Delta^{\bullet}_A(e) \dot{\otimes}^{\boldsymbol{L}}_{A} T^{\vee \bullet} \\ &\cong T^{\vee \bullet} \dot{\otimes}^{\boldsymbol{L}}_{B} \Delta^{\bullet}_B(f) \dot{\otimes}^{\boldsymbol{L}}_{B} T^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_{A} T^{\vee \bullet} \\ &\cong T^{\vee \bullet} \dot{\otimes}^{\boldsymbol{L}}_{B} \Delta^{\bullet}_B(f). \end{aligned}$$

□

Definition 3.10. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For a two-sided recollement tilting complex ${}_B T^{\bullet}_A$ related to idempotents e, f, we define*

$$\Delta^{\bullet}_T = T^{\bullet} \dot{\otimes}^{\boldsymbol{L}}_{A} \Delta^{\bullet}_A(e) \in \mathsf{D}(B^{\circ} \otimes A), \quad \Delta^{\vee \bullet}_T = \Delta^{\bullet}_A(e) \dot{\otimes}^{\boldsymbol{L}}_{A} T^{\vee \bullet} \in \mathsf{D}(A^{\circ} \otimes B).$$

Proposition 3.11. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For a two-sided recollement tilting complex ${}_B T^{\bullet}_A$ related to idempotents e, f, let*

$$\begin{aligned} F' &= \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(\Delta^{\bullet}_T, -) : \mathsf{D}_{A/AeA}(A) \to \mathsf{D}_{B/BfB}(B), \\ F &= \boldsymbol{R}\mathrm{Hom}^{\bullet}_A(T^{\bullet}, -) : \mathsf{D}(A) \to \mathsf{D}(B), \\ F'' &= \boldsymbol{R}\mathrm{Hom}^{\bullet}_{eAe}(fT^{\bullet}e, -) : \mathsf{D}(eAe) \to \mathsf{D}(fBf). \end{aligned}$$

Then the following hold.

(1) *We have an isomorphism* $F' \cong - \dot{\otimes}^{\boldsymbol{L}}_{A} \Delta^{\vee \bullet}_T$.

(2) *A quasi-inverse G' of F' is isomorphic to* $\boldsymbol{R}\mathrm{Hom}^{\bullet}_B(\Delta^{\vee \bullet}_T, -) \cong - \dot{\otimes}^{\boldsymbol{L}}_{B} \Delta^{\bullet}_T$.

(3) *F', F, F'' induce that the recollement $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ is triangle equivalent to the recollement $\{\mathsf{D}_{B/BfB}(B), \mathsf{D}(B), \mathsf{D}(fBf)\}$.*

Proof. According to Proposition 3.2, F' exists and satisfies $F' \cong i_B^{f*} F i_{A*}^e \cong i_B^{f!} F i_{A*}^e$. By Proposition 2.17, we have isomorphisms

$$\begin{aligned} i_B^{f*} F i_{A*}^e &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, -) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_B(f) \\ &\cong - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_B(f), \\ i_B^{f!} F i_{A*}^e &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(\Delta^{\bullet}_B(f), \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, -)) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_B(f) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\bullet}, -). \end{aligned}$$

Let $G = \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(T^{\vee\bullet}, -)$. Since $G' \cong i_A^{e*} G i_{B*}^f \cong i_B^{e!} G i_{B*}^f$, we have isomorphisms

$$\begin{aligned} i_A^{e*} G i_{B*}^f &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(T^{\vee\bullet}, -) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e) \\ &\cong - \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e), \\ i_A^{e!} G i_{B*}^f &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_A(e), \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(T^{\vee\bullet}, -)) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(\Delta^{\bullet}_A(e) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet}, -). \end{aligned}$$

By Corollary 3.9, we complete the proof. ☐

Corollary 3.12. *Under the condition of Proposition 3.11, the following hold.*

1. *$Res_A \Delta^{\bullet}_T$ is a compact object in $\mathsf{D}_{A/AeA}(A)$.*
2. *$Res_{B^\circ} \Delta^{\bullet}_T$ is a compact object in $\mathsf{D}_{(B/BfB)^\circ}(B^\circ)$.*
3. *$\boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_T, -) : \mathsf{D}^*_{A/AeA}(A) \xrightarrow{\sim} \mathsf{D}^*_{B/BfB}(B)$ is a triangle equivalence, where $* =$ nothing, $+, -, \mathrm{b}$.*

Proof. 1, 2. By Corollary 2.9, it is trivial.

3. Since for any $X^{\bullet} \in \mathsf{D}_{A/AeA}(A)$ we have isomorphisms in $\mathsf{D}_{B/BfB}(B)$:

$$\begin{aligned} F'(X^{\bullet}) &= \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_T, X^{\bullet}) \\ &= \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e), X^{\bullet}) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_A(e), X^{\bullet})) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, X^{\bullet}), \end{aligned}$$

we have $\operatorname{Im} F'|_{\mathsf{D}^*_{A/AeA}(A)} \subset \mathsf{D}^*_{B/BfB}(B)$, where $* =$ nothing, $+, -, \mathrm{b}$. Let $G' = \boldsymbol{R}\operatorname{Hom}^{\bullet}_B(\Delta^{\vee\bullet}_T, -)$, then we have also $\operatorname{Im} G'|_{\mathsf{D}^*_{B/BfB}(B)} \subset \mathsf{D}^*_{A/AeA}(A)$, where $* =$ nothing, $+, -, \mathrm{b}$. Since G' is a quasi-inverse of F', we complete the proof. ☐

Proposition 3.13. *Let A, B be k-projective algebras over a commutative ring k, and e, f idempotents of A, B, respectively. For a two-sided recollement tilting complex ${}_B T^{\bullet}_A$ related to idempotents e, f, the following hold.*

1. *$\boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_T, \Delta^{\bullet}_T) \cong \Delta^{\bullet}_T \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\vee\bullet}_T \cong \Delta^{\bullet}_B(f)$ in $\mathsf{D}(B^{\mathrm{e}})$.*
2. *$\boldsymbol{R}\operatorname{Hom}^{\bullet}_{B^\circ}(\Delta^{\bullet}_T, \Delta^{\bullet}_T) \cong \Delta^{\vee\bullet}_T \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_T \cong \Delta^{\bullet}_A(e)$ in $\mathsf{D}(A^{\mathrm{e}})$.*
3. *We have a ring isomorphism $\operatorname{End}_{\mathsf{D}(A)}(\Delta^{\bullet}_T) \cong B/BfB$.*
4. *We have a ring isomorphism $\operatorname{End}_{\mathsf{D}(B^\circ)}(\Delta^{\bullet}_T) \cong (A/AeA)^\circ$.*

Proof. 1. By Corollaries 2.19, 3.9, Proposition 3.11, we have isomorphisms in $\mathsf{D}(B^{\mathrm{e}})$:

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\bullet}_A(\Delta^{\bullet}_T, \Delta^{\bullet}_T) &\cong \Delta^{\bullet}_T \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\vee\bullet}_T \\ &\cong \Delta^{\bullet}_B(f) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_B(f) \\ &\cong \Delta^{\bullet}_B(f) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_B(f) \\ &\cong \Delta^{\bullet}_B(f).\end{aligned}$$

2. By Remark 2.20, Corollary 2.19, we have isomorphisms in $\mathsf{D}(A^{\mathrm{e}})$:

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\bullet}_{B^{\circ}}(\Delta^{\bullet}_T, \Delta^{\bullet}_T) &= \boldsymbol{R}\operatorname{Hom}^{\bullet}_{B^{\circ}}(T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e), T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e)) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_{A^{\circ}}(\Delta^{\bullet}_A(e), \boldsymbol{R}\operatorname{Hom}^{\bullet}_{B^{\circ}}(T^{\bullet}, T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e))) \\ &\cong \boldsymbol{R}\operatorname{Hom}^{\bullet}_{A^{\circ}}(\Delta^{\bullet}_A(e), \Delta^{\bullet}_A(e)) \\ &\cong \Delta^{\bullet}_A(e),\end{aligned}$$

and have isomorphisms in $\mathsf{D}(A^{\mathrm{e}})$:

$$\begin{aligned}\Delta^{\vee\bullet}_T \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B \Delta^{\bullet}_T &\cong \Delta^{\bullet}_A(e) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e) \\ &\cong \Delta^{\bullet}_A(e) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e) \\ &\cong \Delta^{\bullet}_A(e).\end{aligned}$$

3. By Corollaries 2.19, 3.9, we have ring isomorphisms:

$$\begin{aligned}\operatorname{End}_{\mathsf{D}(A)}(\Delta^{\bullet}_T) &\cong \operatorname{End}_{\mathsf{D}(B)}(\Delta^{\bullet}_T \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet}) \\ &\cong \operatorname{End}_{\mathsf{D}(B)}(\Delta^{\bullet}_B(f) \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_B T^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\bullet}) \\ &\cong \operatorname{End}_{\mathsf{D}(B)}(\Delta^{\bullet}_B(f)) \\ &\cong B/BfB.\end{aligned}$$

4. By taking cohomology of the isomorphism of 2, we have the statement by Remark 2.20. □

We give some tilting complexes satisfying the following proposition in Section 4.

Proposition 3.14. *Let A, B be k-projective algebras over a commutative ring k, e an idempotent of A, $P^{\bullet}$ a recollement tilting complex related to e, and $B \cong \operatorname{End}_{\mathsf{D}(A)}(P^{\bullet})$. If $P^{\bullet} \overset{\bullet}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\bullet}_A(e) \cong \Delta^{\bullet}_A(e)$ in $\mathsf{D}(A)$, then the following hold.*

(1) *$A/AeA \cong B/BfB$ as a ring, where f is an idempotent of B corresponding to e.*
(2) *The standard equivalence $\boldsymbol{R}\operatorname{Hom}^{\bullet}_A(T^{\bullet}, -) : \mathsf{D}(A) \to \mathsf{D}(B)$ induces an equivalence $R^0\operatorname{Hom}^{\bullet}_A(T^{\bullet}, -)|_{\mathsf{Mod}\,A/AeA} : \mathsf{Mod}\,A/AeA \to \mathsf{Mod}\,B/BfB$, where ${}_BT^{\bullet}_A$ is the associated two-sided tilting complex of $P^{\bullet}$.*

Proof. 1. By the assumption, we have an isomorphism $Res_A\Delta^{\bullet}_T \cong Res_A\Delta^{\bullet}_A(e)$ in $\mathsf{D}(A)$. By Corollary 2.19, Proposition 3.13, we have the statement.

2. Let $\mathsf{D}^0_{A/AeA}(A)$ (resp., $\mathsf{D}^0_{B/BfB}(B)$) be the full subcategory of $\mathsf{D}_{A/AeA}(A)$ (resp., $\mathsf{D}_{B/BfB}(B)$) consisting of complexes $X^{\bullet}$ with $\mathrm{H}^i(X^{\bullet}) = 0$ for $i \neq 0$. This

category is equivalent to $\mathsf{Mod}\, A/AeA$ (res., $\mathsf{Mod}\, B/BfB$). By Corollary 3.9, we have isomorphisms in $\mathsf{D}(B)$:

$$\begin{aligned}\Delta_T^{\vee\cdot} &\cong \Delta^{\cdot}_A(e) \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\cdot} \\ &\cong T^{\cdot} \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_A \Delta^{\cdot}_A(e) \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\cdot} \\ &\cong \Delta^{\cdot}_B(f) \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_B T^{\cdot} \overset{\cdot}{\otimes}{}^{\boldsymbol{L}}_A T^{\vee\cdot} \\ &\cong \Delta^{\cdot}_B(f).\end{aligned}$$

Define

$$F' = \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_T, -) : \mathsf{D}_{A/AeA}(A) \to \mathsf{D}_{B/BfB}(B),$$
$$G' = \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\vee\cdot}_T, -) : \mathsf{D}_{B/BfB}(B) \to \mathsf{D}_{A/AeA}(A),$$

then they induce an equivalence between $\mathsf{D}_{A/AeA}(A)$ and $\mathsf{D}_{B/BfB}(B)$, by Proposition 3.11. For any $X \in \mathsf{Mod}\, A/AeA$, we have isomorphisms in $\mathsf{D}(k)$:

$$\begin{aligned}Res_k\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_T, X) &\cong Res_k\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_A(e), X) \\ &\cong X.\end{aligned}$$

This means that $\operatorname{Im} F'|_{\mathsf{Mod}\, A/AeA}$ is contained in $\mathsf{D}^0_{B/BfB}(B)$. Similarly since we have isomorphisms in $\mathsf{D}(k)$:

$$\begin{aligned}Res_k\boldsymbol{R}\operatorname{Hom}^{\cdot}_B(\Delta^{\vee\cdot}_T, Y) &\cong Res_k\boldsymbol{R}\operatorname{Hom}^{\cdot}_B(\Delta^{\cdot}_B(f), Y) \\ &\cong Y,\end{aligned}$$

for any $Y \in \mathsf{Mod}\, B/BfB$, $\operatorname{Im} G'|_{\mathsf{Mod}\, B/BfB}$ is contained in $\mathsf{D}^0_{A/AeA}(A)$. Therefore F' and G' induce an equivalence between $\mathsf{D}^0_{A/AeA}(A)$ and $\mathsf{D}^0_{B/BfB}(B)$. Since we have isomorphisms in $\mathsf{D}(B)$:

$$\begin{aligned}\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(T^{\cdot}, X) &\cong \boldsymbol{R}\operatorname{Hom}^{\cdot}_A(T^{\cdot}, i^e_{A*}(X)) \\ &\cong i^f_{B*}\boldsymbol{R}\operatorname{Hom}^{\cdot}_A(\Delta^{\cdot}_T, X)\end{aligned}$$

for any $X \in \mathsf{Mod}\, A/AeA$, we complete the proof. □

4. Tilting Complexes over symmetric Algebras

Throughout this section, A is a finite dimensional algebra over a field k, and $D = \operatorname{Hom}_k(-, k)$. A is called a symmetric k-algebra if $A \cong DA$ as A-bimodules. In the case of symmetric algebras, the following basic property has been seen in [18].

Lemma 4.1. *Let A be a symmetric algebra over a field k, and $P^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. For a bounded complex $X^{\cdot}$ of finitely generated right A-modules, we have an isomorphism:*

$$\operatorname{Hom}^{\cdot}_A(P^{\cdot}, X^{\cdot}) \cong D\operatorname{Hom}^{\cdot}_A(X^{\cdot}, P^{\cdot}).$$

In particular we have an isomorphism:

$$\operatorname{Hom}_{\mathsf{K}(A)}(P^{\cdot}, X^{\cdot}[n]) \cong D\operatorname{Hom}_{\mathsf{K}(A)}(X^{\cdot}, P^{\cdot}[-n])$$

for any $n \in \mathbb{Z}$.

Definition 4.2. *For a complex $X^{\cdot}$, we denote $l(X^{\cdot}) = \max\{n \mid \mathrm{H}^n(X^{\cdot}) \neq 0\} - \min\{n \mid \mathrm{H}^n(X^{\cdot}) \neq 0\} + 1$. We call $l(X^{\cdot})$ the length of a complex $X^{\cdot}$.*

We redefine precisely Definition 2.12 for constructing tilting complexes.

Definition 4.3. *Let A be a finite dimensional algebra over a field k, M a finitely generated A-module, and $P^\bullet : P^{s-r} \to \dots \to P^{s-1} \to P^s \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ a partial tilting complex of length $r+1$. For an integer $n \geq 0$, by induction, we construct a family $\{\Delta^\bullet_n(P^\bullet, M)\}_{n\geq 0}$ of complexes as follows.*

Let $\Delta^\bullet_0(P^\bullet, M) = M$. For $n \geq 1$, by induction we construct a triangle $\zeta_n(P^\bullet, M)$:

$$P^\bullet_n[n+s-r-1] \xrightarrow{g_n} \Delta^\bullet_{n-1}(P^\bullet, M) \xrightarrow{h_n} \Delta^\bullet_n(P^\bullet, M) \to P^\bullet_n[n+s-r]$$

as follows. If $\mathrm{Hom}_{\mathsf{K}(A)}(P^\bullet, \Delta^\bullet_{n-1}(P^\bullet, M)[r-s-n+1]) = 0$, then we set $P^\bullet_n = 0$. Otherwise, we take $P^\bullet_n \in \mathsf{add}\, P^\bullet$ and a morphism $g'_n : P^\bullet_n \to \Delta^\bullet_{n-1}(P^\bullet, M)[r-s-n+1]$ such that $\mathrm{Hom}_{\mathsf{K}(A)}(P^\bullet, g'_n)$ is a projective cover as $\mathrm{End}_{\mathsf{D}(A)}(P^\bullet)$-modules, and $g_n = g'_n[n+s-r-1]$. Moreover, $\Delta^\bullet_\infty(P^\bullet, M) = \underrightarrow{\mathsf{hocolim}}\Delta^\bullet_n(P^\bullet, M)$ and $\Theta^\bullet_n(P^\bullet, M) = \Delta^\bullet_n(P^\bullet, M) \oplus P^\bullet[n+s-r]$.

By the construction, we have the following properties.

Lemma 4.4. *For $\{\Delta^\bullet_n(P^\bullet, M)\}_{n\geq 0}$, we have isomorphisms:*

$$\mathrm{H}^{r-n+i}(\Delta^\bullet_n(P^\bullet, M)) \cong \mathrm{H}^{r-n+i}(\Delta^\bullet_{n+j}(P^\bullet, M))$$

for all $i > 0$ and $\infty \geq j \geq 0$.

Lemma 4.5. *For $\{\Delta^\bullet_n(P^\bullet, M)\}_{n\geq 0}$ and $\infty \geq n \geq r$, we have*

$$\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet, \Delta^\bullet_n(P^\bullet, M)[i]) = 0$$

for all $i \neq r-n-s$.

Proof. Applying $\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet, -)$ to $\zeta_n(P^\bullet, M)$ $(n \geq 1)$, in case of $0 \leq n \leq r$ we have

$$\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet[s], \Delta^\bullet_n(P^\bullet, M)[i]) = 0$$

for $i > r-n$ or $i < 0$. Then in case of $n \geq r$ we have

$$\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet, \Delta^\bullet_n(P^\bullet, M)[i]) = 0$$

for $i \neq r-n-s$. □

Theorem 4.6. *Let A be a symmetric algebra over a field k, and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ a partial tilting complex of length $r+1$. Then the following are equivalent.*

(1) $\mathrm{H}^i(\Delta^\bullet_r(P^\bullet, A)) = 0$ *for all $i > 0$.*
(2) $\Theta^\bullet_n(P^\bullet, A)$ *is a tilting complex for any $n \geq r$.*

Proof. According to the construction of $\Delta^\bullet_n(P^\bullet, A)$, it is clear that $\Theta^\bullet_n(P^\bullet, A)$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. By Lemmas 4.1 and 4.5, it is easy to see that $\Theta^\bullet_n(P^\bullet, A)$ is a tilting complex for A if and only if $\mathrm{Hom}_{\mathsf{D}(A)}(\Delta^\bullet_n(P^\bullet, A), \Delta^\bullet_n(P^\bullet, A)[i]) = 0$ for all $i > 0$. By Lemma 4.4, we have

$$\begin{aligned}\mathrm{H}^i(\Delta^\bullet_r(P^\bullet, A)) &\cong \mathrm{H}^i(\Delta^\bullet_n(P^\bullet, A)) \\ &\cong \mathrm{Hom}_{\mathsf{D}(A)}(A, \Delta^\bullet_n(P^\bullet, A)[i])\end{aligned}$$

for all $i > 0$. For $j \leq n$, applying $\mathrm{Hom}_{\mathsf{D}(A)}(-, \Delta^\bullet_n(P^\bullet, A))$ to $\zeta_j(P^\bullet, A)$, we have

$$\mathrm{Hom}_{\mathsf{D}(A)}(\Delta^\bullet_j(P^\bullet, A), \Delta^\bullet_n(P^\bullet, A)[i]) \cong \mathrm{Hom}_{\mathsf{D}(A)}(\Delta^\bullet_{j-1}(P^\bullet, A), \Delta^\bullet_n(P^\bullet, A)[i])$$

for all $i > 0$, because $\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet[j+s-r-1], \Delta^\bullet_n(P^\bullet, A)[i]) = 0$ for all $i \geq 0$. Therefore $\mathrm{Hom}_{\mathsf{D}(A)}(A, \Delta^\bullet_n(P^\bullet, A)[i]) = 0$ for all $i > 0$ if and only if $\mathrm{Hom}_{\mathsf{D}(A)}(\Delta^\bullet_n(P^\bullet, A), \Delta^\bullet_n(P^\bullet, A)[i]) = 0$ for all $i > 0$. □

Corollary 4.7. *Let A be a symmetric algebra over a field k, $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ a partial tilting complex of length $r+1$, and $V^\bullet$ the associated bimodule complex of $P^\bullet$. Then the following are equivalent.*

(1) $\mathrm{H}^i(\Delta^\bullet_A(V^\bullet)) = 0$ *for all* $i > 0$.
(2) $\Theta^\bullet_n(P^\bullet, A)$ *is a tilting complex for any* $n \geq r$.

Proof. According to Corollary 2.16, we have $\Delta^\bullet_A(V^\bullet) \cong \Delta^\bullet_\infty(P^\bullet, A)$ in $\mathsf{D}(A)$. Since $\mathrm{H}^i(\Delta^\bullet_\infty(P^\bullet, A)) \cong \mathrm{H}^i(\Delta^\bullet_r(P^\bullet, A))$ for $i > 0$, we complete the proof by Theorem 4.6. □

In the case of symmetric algebras, we have a complex version of extensions of classical partial tilting modules which was showed by Bongartz [3].

Corollary 4.8. *Let A be a symmetric algebra over a field k, and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ a partial tilting complex of length 2. Then $\Theta^\bullet_n(P^\bullet, A)$ is a tilting complex for any $n \geq 1$.*

Proof. By the construction, $\Delta^i_1(P^\bullet, A) = 0$ for $i > 0$. According to Theorem 4.6 we complete the proof. □

For an object M in an additive category, we denote by $n(M)$ the number of indecomposable types in $\mathsf{add}\,M$.

Corollary 4.9. *Let A be a symmetric algebra over a field k, and $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ a partial tilting complex of length 2. Then the following are equivalent.*

(1) $P^\bullet$ *is a tilting complex for* A.
(2) $n(P^\bullet) = n(A)$.

Proof. We may assume $P^\bullet : P^{-1} \to P^0$. Since $\Theta^\bullet_1(P^\bullet, A) = P^\bullet \oplus \Delta^\bullet_1(P^\bullet, A)$, by Corollary 4.8, we have $n(A) = n(\Theta^\bullet_1(P^\bullet, A)) = n(P^\bullet) + m$ for some $m \geq 0$. It is easy to see that $m = 0$ if and only if $\mathsf{add}\,\Theta^\bullet_1(P^\bullet, A) = \mathsf{add}\,P^\bullet$. □

Lemma 4.10. *Let $\theta : \mathbf{1}_{\mathsf{D}(eAe)} \to j^{e*}_A j^e_{A!}$ be the adjunction arrow, and let $X^\bullet \in \mathsf{D}(eAe)$ and $Y^\bullet \in \mathsf{D}(A)$. For $h \in \mathrm{Hom}_{\mathsf{D}(A)}(j^e_{A!}(X^\bullet), Y^\bullet)$, let $\Phi(h) = j^{e*}_A(h) \circ \theta_X$, then $\Phi : \mathrm{Hom}_{\mathsf{D}(A)}(j^e_{A!}(X^\bullet), Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(A)}(X^\bullet, j^{e*}_A Y^\bullet)$ is an isomorphism as $\mathrm{End}_{\mathsf{D}(A)}(X^\bullet)$-modules.*

Theorem 4.11. *Let A be a symmetric algebra over a field k, e an idempotent of A, $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,eAe)$ a tilting complex for eAe, and $P^\bullet = j^e_{A!}(Q^\bullet) \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ with $l(P^\bullet) = r+1$. For $n \geq r$, the following hold.*

(1) $\Theta^\bullet_n(P^\bullet, A)$ *is a recollement tilting complex related to* e.
(2) $A/AeA \cong B/BfB$, *where* $B = \mathrm{End}_{\mathsf{D}(A)}(\Theta^\bullet_n(P^\bullet, A))$ *and* f *is an idempotent of* B *corresponding to* e.

Proof. We may assume $P^\bullet : P^{-r} \to \dots P^{-1} \to P^0$. Since $j^e_{A!}$ is fully faithful, $\mathrm{Hom}_{\mathsf{D}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Consider a family $\{\Delta^\bullet_n(P^\bullet, A)\}_{n \geq 0}$ of Definition 4.3 and triangles $\zeta_n(P^\bullet, A)$:

$$P^\bullet_n[n-r-1] \xrightarrow{g_n} \Delta^\bullet_{n-1}(P^\bullet, A) \xrightarrow{h_n} \Delta^\bullet_n(P^\bullet, A) \to P^\bullet_n[n-r].$$

The morphism Φ of Lemma 4.10 induces isomorphisms between exact sequences in $\mathsf{Mod}\, B$:

$$\begin{array}{cc} \mathrm{Hom}_{\mathsf{D}(A)}(P^{\cdot}, P^{\cdot}_n[n-r-1+i]) \to & \mathrm{Hom}_{\mathsf{D}(A)}(P^{\cdot}, \Delta^{\cdot}_{n-1}(P^{\cdot}, A)[i]) \to \\ \downarrow \Phi & \downarrow \Phi \\ \mathrm{Hom}_{\mathsf{D}(eAe)}(Q^{\cdot}, j^{e*}_A P^{\cdot}_n[n-r-1+i]) \to & \mathrm{Hom}_{\mathsf{D}(eAe)}(Q^{\cdot}, j^{e*}_A \Delta^{\cdot}_{n-1}(P^{\cdot}, A)[i]) \to \end{array}$$

$$\begin{array}{cc} \mathrm{Hom}_{\mathsf{D}(A)}(P^{\cdot}, \Delta^{\cdot}_n(P^{\cdot}, A)[i]) \to & \mathrm{Hom}_{\mathsf{D}(A)}(P^{\cdot}, P^{\cdot}_n[n-r+i]) \\ \downarrow \Phi & \downarrow \Phi \\ \mathrm{Hom}_{\mathsf{D}(eAe)}(Q^{\cdot}, j^{e*}_A \Delta^{\cdot}_n(P^{\cdot}, A)[i]) \to & \mathrm{Hom}_{\mathsf{D}(eAe)}(Q^{\cdot}, j^{e*}_A P^{\cdot}_n[n-r+i]) \end{array}$$

for all i. By Lemma 4.10, we have $j^{e*}_A(\zeta_n(P^{\cdot}, A)) \cong \zeta_n(Q^{\cdot}, j^{e*}_A A)$ in $\mathsf{D}(eAe)$, and then $\{j^{e*}_A(\Delta^{\cdot}_n(P^{\cdot}, A))\}_{n\geq 0} \cong \{\Delta^{\cdot}_n(Q^{\cdot}, Ae)\}_{n\geq 0}$. By lemma 4.5, it is easy to see that

$$\mathrm{Hom}_{\mathsf{D}(eAe)}(Q^{\cdot}, \Delta^{\cdot}_\infty(Q^{\cdot}, Ae)[i]) = 0$$

for all $i \in \mathbb{Z}$. Since $Q^{\cdot}$ is a tilting complex for eAe, $\Delta^{\cdot}_\infty(Q^{\cdot}, Ae)$ is a null complex, that is $\mathrm{H}^i(\Delta^{\cdot}_\infty(Q^{\cdot}, Ae)) = 0$ for all $i \in \mathbb{Z}$. By Lemma 4.4, for $n \geq r$ we have $\mathrm{H}^i(\Delta^{\cdot}_n(Q^{\cdot}, Ae)) = 0$ for all $i > 0$. By the above isomorphism, for $n \geq r$ we have $\mathrm{H}^i(\Delta^{\cdot}_n(P^{\cdot}, A)) \in \mathsf{Mod}\, A/AeA$ for all $i > 0$. On the other hand, $\Delta^{\cdot}_n(P^{\cdot}, A)$ has the form:

$$R^{\cdot} : R^{-n} \to \ldots \to R^0 \to R^1 \to \ldots \to R^{r-1},$$

where $R^i \in \mathsf{add}\, eA$ for $i \neq 0$, and $R^0 = A \oplus R'^0$ with $R'^0 \in \mathsf{add}\, eA$. Since $\mathrm{Hom}_A(eA, \mathsf{Mod}\, A/AeA) = 0$, it is easy to see that $\Delta^{\cdot}_n(P^{\cdot}, A) \cong \sigma_{\leq 0}\Delta^{\cdot}_n(P^{\cdot}, A)$ ($\cong \sigma_{\leq 0} \ldots \sigma_{\leq r-2}\Delta^{\cdot}_n(P^{\cdot}, A)$ if $r \geq 2$). Therefore, $\mathrm{H}^i(\Delta^{\cdot}_n(P^{\cdot}, A)) = 0$ for all $i > 0$, and hence $\Theta^{\cdot}_n(P^{\cdot}, A)$ is a recollement tilting complex related to e by Theorem 4.6. Since $\Theta^{\cdot}_n(P^{\cdot}, A) \cong P^{\cdot}[n-r] \oplus R^{\cdot}$ and $j^e_{A!}(X^{\cdot}) \overset{\boldsymbol{L}}{\dot{\otimes}}_A \Delta^{\cdot}_A(e) = i^{e*}_A j^e_{A!}(X^{\cdot}) = 0$ for $X^{\cdot} \in \mathsf{D}(eAe)$, we have an isomorphism $\Theta^{\cdot}_n(P^{\cdot}, A) \dot{\otimes}_A \Delta^{\cdot}_A(e) \cong \Delta^{\cdot}_A(e)$ in $\mathsf{D}(A)$. By Proposition 3.14, we complete the proof. □

Corollary 4.12. *Under the condition Theorem 4.11, let ${}_BT^{\cdot}_A$ be the associated two-sided tilting complex of $\Theta^{\cdot}_n(P^{\cdot}, A)$. Then the standard equivalence $\boldsymbol{R}\mathrm{Hom}^{\cdot}_A(T^{\cdot}, -) : \mathsf{D}(A) \xrightarrow{\sim} \mathsf{D}(B)$ induces an equivalence $R^0\mathrm{Hom}^{\cdot}_A(T^{\cdot}, -)|_{\mathsf{Mod}\, A/AeA} : \mathsf{Mod}\, A/AeA \xrightarrow{\sim} \mathsf{Mod}\, B/BfB$.*

Proof. By the proof of Theorem 4.11, we have $T^{\cdot} \overset{\boldsymbol{L}}{\dot{\otimes}}_A \Delta^{\cdot}_A(e) \cong \Delta^{\cdot}_A(e)$ in $\mathsf{D}(A)$. By Proposition 3.14, we complete the proof. □

Remark 4.13. *For a symmetric algebra A over a field k and an idempotent e of A, eAe is also a symmetric k-algebra. Therefore, we have constructions of tilting complexes with respect to any sequence of idempotents of A. Moreover, if a recollement $\{\mathsf{D}_{A/AeA}(A), \mathsf{D}(A), \mathsf{D}(eAe)\}$ is triangle equivalent to a recollement $\{\mathsf{D}_{B/BfB}(B), \mathsf{D}(B), \mathsf{D}(fBf)\}$, then B and fBf are also symmetric k-algebras.*

Remark 4.14. *According to* [17], *under the condition of Theorem 4.11 we have a stable equivalence $\underline{\mathsf{mod}}A \xrightarrow{\sim} \underline{\mathsf{mod}}B$ which sends A/AeA-modules to B/BfB-modules, where $\underline{\mathsf{mod}}A, \underline{\mathsf{mod}}B$ are stable categories of finitely generated modules. In particular, this equivalence sends simple A/AeA-modules to simple B/BfB-modules.*

Remark 4.15. *Let A be a ring, and e an idempotent of A such that there is a finitely generated projective resolution of Ae in $\mathsf{Mod}\, eAe$. Then Hoshino and Kato showed that $\Theta^{\cdot}_n(eA, A)$ is a tilting complex if and only if $\mathrm{Ext}^i_A(A/AeA, eA) = 0$*

for $0 \leq i < n$ *([7]). In even this case, we have also* $A/AeA \cong B/BfB$*, where* $B = \mathrm{End}_{\mathsf{D}(A)}(\Theta^{\bullet}_n(eA, A))$ *and* f *is an idempotent of* B *corresponding to* e*. Moreover if* A*,* B *are* k*-projective algebras over a commutative ring* k*, then by Proposition 3.14 the standard equivalence induces an equivalence* $\mathsf{Mod}\, A/AeA \xrightarrow{\sim} \mathsf{Mod}\, B/BfB$*.*

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